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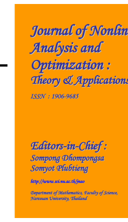


# ON GENERALIZED VARIATIONAL INEQUALITY PROBLEMS◇

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**ABSTRACT.** In this paper, we introduce and study the generalized implicit vector variational inequality problems with set valued mappings in topological vector spaces. We establish existence theorems for the solution set of these problems to be nonempty compact and convex. Our results extend the results by Fang and Huang [Existence results for generalized implicit vector variational inequalities with multivalued mappings, Indian J. Pure and Appl. Math. 36(2005), 629-640].

**KEYWORDS** : Implicit vector variational inequality; Set valued mapping; Affine mapping,
$C$-pseudomonotone; Strongly $C$-pseudomonotone.

## 1. INTRODUCTION

The vector variational inequality, as an important generalization of the scalar variational inequality, has been shown to have wide applications to vector optimization problems and vector equilibrium problems (see [1, 2, 4, 10]). The first result on the vector variational inequality is paper [8] by Giannessi, which studies the vector variational inequality in the setting of a finite dimensional case. Later on, Chen and Yang [4] studied the vector variational inequalities in infinite dimensional spaces. The theory of vector variational inequalities can be used to the study vector complementarity problems and multi-objective programming problems. For details, we refer to [1, 2, 8, 9, 10].

Throughout this paper, unless otherwise specified, we always let $X$ and $Y$ be real Hausdorff topological vector spaces, $K \subseteq X$ a nonempty convex set, $C : K \longrightarrow 2^Y$ with pointed closed cone convex values ( we recall that a subset $A$ of $Y$ is convex

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cone and pointed whenever $A + A \subseteq Y, tA \subseteq A$, for $t \geq 0$, and $A \cap -A = \{0\}$ respectively) where $2^Y$ denotes all the subsets of $Y$. Denote by $L(X, Y)$ the set of all continuous linear mappings from $X$ into $Y$. For any given $l \in L(X, Y),\ x \in X$, let $\langle l, x\rangle$ denote the value of $l$ at $x$. Let $T : K \longrightarrow 2^{L(X,Y)}$ and $G : X \times Y \longrightarrow X$ be two mappings. Finally let $A : K \times K \longrightarrow 2^{L(X,Y)}$ be a set-valued mapping. We need the following definitions and results in the sequel.

**Definition 1.1.** Let $X$ and $Y$ be two topological spaces. A set-valued mapping $G : X \longrightarrow 2^Y$ is called:

(i) **upper semi-continuous** (u.s.c.) at $x \in X$ if for each open set $V$ containing $G(x)$, there is an open set $U$ containing $x$ such that for each $t \in U,\ G(t) \subseteq V;\ G$ is said to be u.s.c. on $X$ if it is u.s.c. at all $x \in X$.

(ii) **upper hemicontinuous** if the restriction of $G$ on straight lines is upper semi-continuous.

(iii) **lower semi-continuous** (l.s.c.) at $x \in X$ if for each open set $V$ with $G(x) \cap V \neq \emptyset$, there is an open set $U$ containing $x$ such that for each $t \in U,\ G(t) \cap V \neq \emptyset;\ G$ is said to be l.s.c. on $X$ if it is l.s.c. at all $x \in X$.

(iv) **closed** if the graph of $G$, i.e., the set $\{(x, y) : x \in X, y \in G(x)\}$, is a closed set in $X \times Y$.

(v) **compact** if the closure of range $G$, i.e., $cl\ G(X)$, is compact, where $G(X) = \bigcup_{x\in X} G(x)$.

(vi) **continuous** if $G$ is both lower semi-continuous and upper semi-continuous.

**Lemma 1.2.** *(([10],[12]). Let $X$ and $Y$ be two topological spaces. Suppose that $G : X \longrightarrow 2^Y$, is a set-valued mapping. Then the following statements are true:*

(a) *If $G$ is closed and compact, then $G$ is u.s.c.*
(b) *If for any $x \in X,\ G(x)$ is compact, then $G$ is u.s.c. on $X$ if and only if for any net $\{x_\alpha\} \subset X$ such that $x_\alpha \longrightarrow x$ and for every $y_\alpha \in G(x_\alpha)$, there exist $y \in G(x)$ and a subnet $\{y_\beta\}$ of $\{y_\alpha\}$ such that $y_\beta \longrightarrow y$.*
(c) *If $G$ is lower semicontinuous then for any closed $C \subseteq Y$ any net $\{x_\alpha\} \subseteq X$ converges to $x$ and $G(x_\alpha) \subset C$ for all $\alpha$ imply that $G(x) \subseteq C$.*

**Definition 1.3.** We say that the mapping $G : K \longrightarrow 2^Y$ is $C$-upper sign-continuous if, for all $x, y \in K$, the following implication holds:

$$G((1-t)x + ty) \cap C((1-t)x + ty) \neq \emptyset,\ \forall t \in ]0, 1[ \Rightarrow G(x) \cap C(x) \neq \emptyset.$$

**Remark 1.4.** Let $f : K \times K \longrightarrow \Re$ be a real mapping. If we define $G(x) = \{f(x, y)\}$, for all $x, y \in K$, and $C(x) = [0, \infty)$, then Definition 1.3 reduces to the upper sign-continuous introduced by Bianchi and Pini in [1]. The upper sign continuity notion was first introduced by Hadjisavvas [10] for a single valued mapping in the framework of variational inequality problems. It is clear that if $G$ is $C$-upper sign-continuous then $G$ is upper hemicontinuous but the simple example $G : \Re \longrightarrow 2^{\Re}$ defined by

$$G(x) = \begin{cases} \{0\} & \text{if } x = 0 \\ \Re & \text{if } x \neq 0 \end{cases}$$

and $C(x) = [0, \infty)$, for $x \in \Re$, shows that the converse dose not hold in general. We note that the map $G$ is upper semicontinuous at each nonzero element of real numbers.

**Definition 1.5.** Let $E$ be a topological vector space. A mapping $F : M \subseteq E \longrightarrow 2^E$ is said to be a KKM mapping, if, for any finite set $A \subseteq M$,

$$\text{co}A \subseteq F(A),$$

where co$A$ denotes the convex hull of $A$.

**Lemma 1.6.** *([5]). Let $M$ be a nonempty subset of a Hausdorff topological vector space $E$ and $F : M \longrightarrow 2^M$ be a KKM mapping. If $F(x)$ is closed in $E$, for every $x \in M$, and compact, for some $x \in M$, then*

$$\bigcap_{x \in M} F(x) \neq \emptyset.$$

**Definition 1.7.** Let $T : K \longrightarrow 2^{L(X,Y)}$ be a set valued mapping. Then $T$ is said to be

(i) strongly $C$-pseudomonotone with respect to $G$ if for any given $x, y \in K$,

$$\langle Tx, G(x, y)\rangle \nsubseteq -\text{int}C(x) \Rightarrow \langle Ty, G(y, x)\rangle \subseteq -\text{int}C(x).$$

(i) $C$-pseudomonotone with respect to $G$ if for any given $x, y \in K$,

$$\langle Tx, G(x, y)\rangle \nsubseteq -C(x) \setminus \{0\} \Rightarrow \langle Ty, G(y, x)\rangle \subseteq -C(y).$$

**Remark 1.8.** (a) It is clear from Definition 1.7 that strongly $C$-pseudomonotone with respect to $G$ implies $C$-pseudomonotone with respect to $G$ but the simple example $X = \Re^2$, $Y = \Re$, $K$ any closed convex subset of $X$, $T(x) = \{(0, 0)\}$, for all $x \in K$ and $G$ is an arbitrary function from $X \times Y$ to $X$ shows that the converse is not valid in general.

(b) If we define $G(x, y) = y - g(x)$ where $g : K \longrightarrow K$ is a mapping, then Definition 1.7 collapses to the Definition 2.3 in [8].

**Example 1.9.** Let $X = K = \Re$, $Y = \Re^2$ and $C(x) = P = \{(x, y) \in \Re^2 : x \geq 0,\ y \geq 0\}$, for all $x, y \in K$. Let us define

$$T(x) = \{\begin{pmatrix} x \\ x^2 \end{pmatrix}\},\ G(x, y) = y - x \text{ and } g(x) = x.$$

Then, obviously, $T(x) \subset L(X, Y)$. If we take $y < x$, $x < 0$, then $\langle T(x), y - x\rangle = \{(y-x)(x, x^2)\} \nsubseteq -\text{int}P$ since $(y-x)x > 0$ and $\langle T(y), y-x\rangle = \{(y-x)(y, y^2)\} \nsubseteq P$ because $(y - x)y^2 < 0$ and so $T$ is not strongly $C$- pseudomonotone mapping with respect to $g$ in the sense of Huang and Fang [8]. While, if $\langle T(y), y-x\rangle \in -\text{int}P$ then $(x - y)(y, y^2) \in \text{int}P$. Thus $x - y > 0$ and $y > 0$ which imply that $\langle T(x), x - y\rangle = (x - y)(x, x^2) \in \text{int}P$ and so $\langle T(x), y - x\rangle \in -\text{int}P$. This shows that $T$ is strongly $C$-pseudomonotone with respect to $G$ in our sense.

## 2. MAIN RESULTS

In this section, we consider the following generalized implicit vector variational inequality problems in the topological vector space setting:

(IVVI$_1$) find $x \in K$ such that $\langle Tx, G(x, y)\rangle \nsubseteq -\text{int}C(x),\ \forall y \in K,$

and

(IVVI$_2$) find $x \in K$ such that $\langle Tx, G(x,y)\rangle \nsubseteq -C(x) \setminus \{0\},\ \forall y \in K.$

Clearly, a solution of problem (IVV$I_2$) is also a solution of problem (IVVI$_1$). Note that if $G(x,y) = 0$, for all $(x,y) \in X \times Y$, then the solution set of (IVVI$_1$) and (IVVI$_2$) are equal to the set $K$. Moreover, if $G(x,y) = 0$, for all $y \in Y$, then $x$ is a solution of (IVVI$_1$) and (IVVI$_2$).

**Example 2.1.** Let $X = \Re,\ K = [1,2],\ Y = \Re^2$ and $C(x) = \{(t,s) \in \Re^2 : 0 \leq s \leq tx\}$, for all $x \in K$. Let $T(x) = \{(t,tx) \in \Re^2 : t \in \Re\}$ and $G(x,y) = 1$, for all $x, y \in K$. One can check that $x = 1$ is a solution (IVVI$_1$) and is not a solution of (IVVI$_2$).

**Example 2.2.** Consider the following linear programming:

$$\min\{\sum_{i=1}^{n} c_i x_i : x = (x_i)_{i=1}^{n} \in \Re^n \text{ with } x \geq 0,\ Ax = b\},$$

where $A$ ( is a matrix), $c = (c_i)_{i=1}^{n} \in \Re^n,\ b \in \Re^m$ are fixed.

Now if we define $X = \Re^n,\ K = \{x \in \Re^n : x \geq 0,\ Ax = b\}$( note $K$ is closed and convex), $Y = \Re,\ C(x) = [0,\infty), G(x,y) = y - x$ and define $T(x) = \{c\}$, for all $x \in K$, then the above linear programming is a special case of (IVVI$_1$).

The following lemmas play a key role in this section. Furthermore, they improve Lemmas 2.3 and 2.4, respectively, of [8]. Precisely, they extend Lemmas 2.3 and 2.4 of [8] from Banach spaces to topological vector spaces, omit the closedness of $C : K \longrightarrow 2^Y$, replace special case $(x,y) \longrightarrow y - g(x)$, where $g : K \longrightarrow K$ is a mapping, with a general mapping $(x,y) \longrightarrow G(x,y)$, and upper sign-continuityupper with hemicontinuity .

**Lemma 2.3.** *Suppose that*

(1) *for each fixed $y$, the mapping $x \longrightarrow \langle Tx, G(x,y)\rangle$ is upper sign-continuous ;*
(2) *$T$ is $C$-pseudomonotone with respect to $G$;*
(3) *$\langle Tx, G(x,x)\rangle \cap C(x) \neq \emptyset$ for every $x \in K$;*
(4) *$G(x,y)$ is affine in the second variable.*

*Then for any given $y \in K$, the following are equivalent:*

(i) *$\langle Ty, G(y,z)\rangle \nsubseteq -C(y) \setminus \{0\},\ \forall z \in K$;*
(ii) *$\langle Tz, G(z,y)\rangle \subseteq -C(z),\ \forall z \in K$*

*Proof.* The fact $(i) \Rightarrow (ii)$ directly follows from the definition of $C$-pseudomonotonicity with respect to $G$. Now let $(ii)$ hold and $z$ be an arbitrary element of $K$. Hence (ii) implies that

$$\langle Tz_t, G(z_t,y)\rangle \subseteq -C(z_t),\ \forall z \in K, \forall t \in ]0,1[, \qquad (2.1)$$

where $z_t = y + t(z - y)$.

We claim that, for all $t \in ]0,1[$,

$$\langle Tz_t, G(z_t,z)\rangle \cap C(z_t) \neq \emptyset, \qquad (2.2)$$

Otherwise, we have, for some $t \in ]0,1[$,

$$\langle Tz_t, G(z_t,z)\rangle \subseteq Y \setminus C(z_t). \qquad (2.3)$$

From (4) , (2.1) and (2.3) we get

$$\langle Tz_t, G(z_t, z_t)\rangle = \langle Tz_t, (1-t)G(z_t, y) + tG(z_t, z)\rangle =$$

$$(1-t)\langle Tz_t, G(z_t, y)\rangle + t\langle Tz_t, G(z_t, z)\rangle \subseteq -C(z_t) + Y \setminus C(z_t) \subseteq Y \setminus C(z_t),$$

which is a contradiction ( with condition (3)). Thus,

$$\langle Tz_t, G(z_t, z)\rangle \cap C(z_t) \neq \emptyset,\ \forall t \in ]0, 1[,$$

and so by (1) we deduce that

$$\langle Ty, G(y, z)\rangle \cap C(y) \neq \emptyset.$$

This completes the proof. □

**Remark 2.4.** (a) We can omit condition (3) of Lemma 2.3 when $G(x, x) = 0, \forall x \in K$. Moreover, we can replace (4) by $C(x)$- convexity of $G(x, y)$ in the second variable, that is, for all $x, z_1, z_2 \in K$ and $t \in ]0, 1[$, the following implication holds:

$$G(x, (1-t)z_1 + tz_2) \subseteq (1-t)G(x, z_1) + tG(x, z_2) - C(x).$$

We note that even in the real line convexity of $G$ with respect second variable is weaker than to be affine $G$ with respect second variable. To see this consider $G(x, y) = y^2$, where $x, y \in \Re$. Hence we obtain another version of Lemma 2.3, for $G(x, x) = 0, \forall x \in K$, as follows:

**Lemma 2.5.** *Suppose that*

(1) *for each fixed $y$, the mapping $x \longrightarrow \langle Tx, G(x, y)\rangle$ is upper sign-continuous ;*
(2) *$T$ is $C$-pseudomonotone with respect to $G$;*
(3) *$G(x, y)$ is $C(x)$-convex in the second variable.*

*Then for any given $y \in K$, the following are equivalent:*

(i) $\langle Ty, G(y, z)\rangle \nsubseteq -C(y) \setminus \{0\},\ \forall z \in K;$
(ii) $\langle Tz, G(z, y)\rangle \subseteq -C(z),\ \forall z \in K.$

We can get, by a similar argument given in lemma 2.5, the following result which is another version of Lemma 2.5 when $G$ is strongly $C$-pseudomonotone in the second variable and $G(x, x) = 0$, for all $x \in K$.

**Lemma 2.6.** *Assume that*

(1) *for each fixed $y \in K$, the mapping $x \longrightarrow \langle Tx, G(x, y)\rangle$ is upper sign-continuous ;*
(2) *$T$ is strongly $C$-pseudomonotone with respect to $G$;*
(3) *$G(x, y)$ is $C(x)$-convex in the second variable.*

*Then for any given $y \in K$, the following are equivalent:*

(i) $\langle Ty, G(y, z)\rangle \nsubseteq -\mathrm{int}\, C(y),\ \forall z \in K;$
(ii) $\langle Tz, G(z, y)\rangle \subseteq -C(z),\ \forall z \in K$

In the following we establish an existence result for (IVV$I_2$) .

**Theorem 2.7.** *Suppose all the assumptions of Lemma 2.3 ( or Lemma 2.5) hold and , for each fixed $x \in K$, the mapping $y \longrightarrow G(x, y)$ is continuous. If there exist a compact convex subset $D$ of $K$ and a compact subset $B$ of $K$ such that*

$$(C) \qquad \forall x \in K \backslash B\ \exists z \in D :\ \langle Tz, G(z, x)\rangle \nsubseteq -C(z),$$

*then the solution set of (IVVI$_2$) is nonempty compact and convex.*

*Proof.* Define $F_1, F_2 : K \longrightarrow 2^K$ by

$$F_1(z) = \{x \in K : \langle Tx, G(x,z)\rangle \nsubseteq -C(x) \setminus \{0\}\},$$

$$F_2(z) = \{x \in K : \langle Tz, G(z,x)\rangle \subseteq -C(z)\}.$$

We claim that $F_1$ is a KKM mapping. If it is not the case, then there exist $z_1, ..., z_n \in K$ and $t_i > 0$ with $\Sigma_{i=1}^n t_i = 1$ such that $x = \Sigma_{i=1}^n t_i z_i \notin \cup_{i=1}^n F_1(z_i)$, i.e.,

$$\langle Tx, G(x, z_i)\rangle \subseteq -C(x) \setminus \{0\},\ i = 1, 2, ..., n.$$

It follows from the condition (4) of Lemma 2.3 ( or condition 3 of Lemma 2.5) that

$$\langle Tx, G(x,x)\rangle = \langle Tx, \Sigma_{i=1}^n t_i G(x, z_i)\rangle \subseteq$$

$$\Sigma_{i=1}^n t_i \langle Tx, \Sigma_{i=1}^n G(x, z_i)\rangle \subseteq -C(x) \setminus \{0\},$$

and so

$$\langle Tx, G(x,x)\rangle \subseteq -C(x) \setminus \{0\}.$$

This and $C(x) \cap (-C(x) \setminus \{0\}) = \emptyset$ ( note that the mapping $C$ has pointed closed cone convex values) imply that

$$\langle Tx, G(x,x) \cap C(x) = \{\emptyset\},$$

which contradicts condition (3) of Lemma 2.3. Hence $F_1$ is a KKM mapping and so $F_2$ is also a KKM mapping ( note, by (2) of Lemma 2.3 we have $F_1(z) \subseteq F_2(z)$ for every $z \in K$). Further, by the continuity of the mapping $y \longrightarrow G(x,y)$ for each fixed $x \in K$, that $F_2(z)$ is closed in $K$, for every $z \in K$. Now $F_2|_{co(A\cup D)}$ ( the restriction of $F_2$ on compact and convex subset $co(A \cup D)$ of $K$ where $A$ is finite subset of $K$) satisfies all the assumptions of Lemma 1.6 and hence

$$\cap_{z \in co(A\cup D)} F_2(z) \neq \emptyset. \qquad (2.4)$$

By condition (C) we have

$$\cap_{z \in D} F_2(z) \subseteq B. \qquad (2.5)$$

From (2.4) and (2.5) we inform that the family $\{F_2(z)\}_{z\in K}$ has finite intersection property and so

$$\cap_{z\in K} F_2(z) \neq \emptyset.$$

Then there exists $x \in K$ such that

$$\langle Tz, G(z,x)\rangle \subseteq -C(z),\ \forall z \in K.$$

Now from Lemma 2.3 ( or Lemma 2.5) we get

$$\langle Tx, G(x,z)\rangle \nsubseteq -C(x) \setminus \{0\},\ \forall z \in K,$$

and so $x$ is a solution of (IVV$I_2$). By Lemma 2.3 ( or Lemma 2.5) the solution set of (IVV$I_2$) equals to the set

$$S = \{x \in K : \langle Tz, G(z,x)\rangle \subseteq -C(z), \forall z \in K\},$$

which is convex by (4) of Lemma 2.3 ( or (3) of Lemma 2.5). Also by our assumption ( that is condition (C)) $S$ is a closed subset of $B$ and hence compact. This completes the proof. □

**Remark 2.8.** (a) One can see, by the definitions of (IVV$I_1$), (IVV$I_2$) and $-int\ C(x) \subseteq -C(x)$, for all $x \in K$, that any solution of (IVV$I_2$)is a solution of (IVV$I_1$). So we can consider Theorem 2.7 as an existence theorem for the solution of (IVV$I_1$).

(b) To be continuity of $G$ in the second variable in Theorem 2.7 can be replaced by the lower semi-continuity of the mapping

$$z \longrightarrow \{x \in K : \langle Tz, G(z,x)\rangle \subseteq -C(z)\}.$$

(c) We can drop condition (C) of Theorem 2.7 when $K$ is compact. Hence Theorem 2.7 improves Lemma 2.7 of [8]. Because, in Lemma 2.4 of [8], the authors supposed that $K$ is a bounded closed subset of a reflexive Banach space $X$ which means $K$ is compact in the $W^*-$ topology on $X$.

The next result guarantees, under suitable conditions, the solution set of (IVV$I_1$) is nonempty compact and convex. We omit its proof, since it is similar to the proof of Theorem 2.7.

**Theorem 2.9.** *Suppose all the assumptions of Lemma 2.6 hold and , for each fixed $x \in K$, the mapping $y \longrightarrow G(x,y)$ is continuous ( or the mapping $z \longrightarrow \{x \in K : \langle Tz, G(z,x)\rangle \subseteq -C(z)\}$ is lower semi-continuous). If there exist a compact convex subset $D$ of $K$ and a compact subset $B$ of $K$ such that*

$$\forall x \in K\backslash B\ \exists z \in D:\ \langle Tz, G(z,x)\rangle \nsubseteq -C(z),$$

*then the solution set of (IVV$I_1$) is nonempty compact and convex*

## 3. APPLICATION

In this section, using Theorems 2.7 and 2.9, we establish some existence theorems for the following two generalized implicit vector variational inequality problems in the locally convex topological vector spaces. Our results extend Theorems 3.1 and 3.2 of [8] from the reflexive Banach spaces to locally convex spaces. Moreover, we do not use the notion demi-C-continuity on our maps. : Now we recall generalized implicit vector variational inequality problems as follow:

find $u \in K$ such that

$$\langle A(u,u), G(u,v)\rangle \nsubseteq -C(u) \setminus \{0\},\ \forall v \in K,$$

and

find $u \in K$ such that

$$\langle A(u,u), G(u,v)\rangle \nsubseteq -\text{int}\ C(u),\ \forall v \in K.$$

In order to prove our existence theorems we need the following result.

**Theorem 3.1.** *( Kakutani-Fan-Glicksberg)([7]). Let $X$ be a locally convex Hausdorff space, $D \subseteq X$ a nonempty, convex compact subset. Let $T : D \longrightarrow 2^D$ be upper semicontinuous with nonempty, closed convex $T(x)$, for all $x \in D$. Then $T$ has a fixed point in $D$.*

**Theorem 3.2.** *Assume that the following conditions hold*

(i) *for each fixed $(w,y) \in K \times K$, the mapping $x \longrightarrow \langle A(w,x), G(x,y)\rangle \cap C(x)$ is upper sign-continuous with compact values;*
(ii) *for each fixed $w \in K$ the mapping $x \longrightarrow A(w,x)$ is $C$-pseudomonotone with respect to $G$;*

(iii) *for each fixed* $w \in K \langle A(w,x), G(x,x)\rangle \cap C(x) \neq \emptyset$;
(iv) $G(x,y)$ *is affine in the second variable;*
(v) *for each finite dimensional subspace* $M$ *of* $X$ *with* $K_M = K \cap M \neq \emptyset$, *there exist compact subset* $B_M$ *and compact convex subset* $D_M$ *of* $K_M$ *such that* $\forall (w,x) \in K_M \times (K_M \backslash B_M)$, $\exists z \in D_M$ *such that* $\langle A(w,z), G(z,x)\rangle \nsubseteq -C(z)$;
(vi) *for each fixed* $v \in K$, *the mapping* $(x,y) \longrightarrow \langle A(x,y), G(v,y)\rangle$ *is lower semicontinuous.*

*Then there exits* $u \in K$ *such that*

$$\langle A(u,u), G(u,v)\rangle \nsubseteq -C(u) \setminus \{0\},\ \forall v \in K.$$

*Proof.* Let $M \subset X$ be a finite dimensional subspace with $K_M = K \cap M \neq \emptyset$. For each fixed $w \in K$, consider the problem of finding $u \in K_M$ such that

$$\langle A(w,u), G(u,v)\rangle \nsubseteq -C(u) \setminus \{0\},\ \forall v \in K_M. \tag{2.6}$$

By Theorem 2.7, the solution set of problem (2.6) is nonempty compact and convex subset of $K_M$ and so the mapping $F : K_M \longrightarrow 2^{K_M}$ defined by

$$F(w) = \{u \in K_M :\ \langle A(w,u), G(u,v)\rangle \nsubseteq -C(u) \setminus \{0\},\ \forall v \in K_M\}$$

has nonempty compact and convex values. Lemma 2.3 implies

$$F(w) = \{u \in K_M :\ \langle A(w,u), G(v,u)\rangle \subseteq -C(v),\ \forall v \in K_M\}.$$

Further, $F$ has closed graph. Indeed, let $(w_\alpha, u_\alpha) \in K_M \times F(w_\alpha)$ converge to $(w,u) \in K_M \times K_M$. Then $\langle A(w_\alpha,u_\alpha), G(v,u_\alpha)\rangle \subseteq -C(v)$, for all $\alpha$ and $v \in K_M$. Now from (vi) we get $\langle A(w,u), G(v,u)\rangle \subseteq -C(v)$ and hence $u \in F(w)$. This shows that the graph of $F$ is closed and so since the values of $F$ are compact we deduce from Lemma 1.2 that $F$ is upper semi-continuous on $K_M$. Therefore, by using the Kakutani-Fan-Glicksberg fixed point theorem (that is Theorem 3.1), $F$ has a fixed point $w_0 \in K_M$, i.e., there exists $w_0 \in K_M$ such that

$$\langle A(w_0,v), G(v,w_0)\rangle \subseteq -C(v),\ \forall v \in K_M.$$

Set $\mathcal{M} = \{M \subset X : M$ is a finite dimensional subspace with $K_M \neq \emptyset\}$ and for $M \in \mathcal{M}$ and

$$W_M = \{u \in K_M\ :\ \langle A(u,v), G(v,u)\rangle \subseteq -C(v),\ \forall v \in K_M\},\ \forall\ M \in \mathcal{M}.$$

Clearly, $W_M$ is nonempty and by (vi),(v)is a compact subset of $B_M$. For each finite subset $\{M_i\}_{i=1}^n$ of $\mathcal{M}$, from the definition of $W_M$, we have $W_{\bigcup_i M_i} \subset \bigcap_{i=1}^n W_{M_i}$, so $\{W_M : M \in \mathcal{M}\}$ has the finite intersection property. Hence, there is $u \in \bigcap_{M\in\mathcal{M}} W_M$. We show that

$$\langle A(u,v), G(v,u)\rangle \subseteq -C(v),\ \forall v \in K.$$

Indeed, for each $v \in K$, there is $M \in \mathcal{M}$ such that $v \in K_M$. Since $W_M$ is closed and $u \in W_M$, there exists a net $\{u_\alpha\} \subset W_M$ such that $u_\alpha$ converges to $u$. It follows that

$$\langle A(u_\alpha,v), G(v,u_\alpha)\rangle \subseteq -C(v).$$

Since $C(v)$ is closed, $G$ is continuous in the second variable, $u_\alpha$ converges to $u$ one has

$$\langle A(u,v), G(v,u)\rangle \subseteq -C(v),\ \forall v \in K.$$

Now Lemma 2.3 implies

$$\langle A(u,u), G(u,v)\rangle \nsubseteq -C(u) \setminus \{0\},\ \forall v \in K.$$

The proof is complete. □

**Remark 3.3.** In Theorem 3.2, we can omit (iii) if $G(x, x) = 0$, for all $x \in K$, and also condition (v) when $K$ is compact. Hence we get Theorem 3.1 of [8] without assuming $K$ is a bounded subset of a reflexive Banach space $X$. Moreover, in Theorem 3.2 $C : K \longrightarrow 2^Y$ does not need to have closed graph as supposed in Theorem 3.1 in [8].

Using Theorem 2.7 and the proof given for Theorem 3.2, we obtain the following theorem.

**Theorem 3.4.** *Assume that the following conditions hold*

(i) *for each fixed* $(w, y) \in K \times K$, *the mapping* $x \longrightarrow \langle A(w, x), G(x, y)\rangle \cap C(x)$ *is upper sign-continuous with compact values;*
(ii) *for each fixed* $w \in K$ *the mapping* $x \longrightarrow A(w, x)$ *is strongly* $C$*-pseudomonotone with respect to* $G$*;*
(iii) *for each fixed* $w \in K$, $\langle A(w, x), G(x, x)\rangle \nsubseteq -intC(x)$;
(iv) $G(x, y)$ *is affine and continuous in the second variable;*
(v) *for each finite dimensional subspace* $M$ *of* $X$ *with* $K_M = K \cap M \neq \emptyset$, *there exist compact subset* $B_M$ *and compact convex subset* $D_M$ *of* $K_M$ *such that* $\forall (w, x) \in K_M \times (K_M \backslash B_M)$, $\exists z \in D_M$ *such that* $\langle A(w, z), G(z, x)\rangle \nsubseteq -C(z)$;
(vi) *for each fixed* $v \in K$, *the mapping* $(x, y) \longrightarrow \langle A(x, y), G(v, y)\rangle$ *is lower semicontinuous.*

*Then there exits* $u \in K$ *such that*

$$\langle A(u, u), G(u, v)\rangle \nsubseteq -\mathit{int}\, C(u),\ \forall v \in K.$$

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