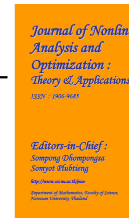

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SOME FIXED POINT RESULTS FOR UNIFORMLY QUASI-LIPSCHITZIAN MAPPINGS IN CONVEX METRIC SPACES

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ABSTRACT. In this paper, an iteration process for approximating common fixed points of two uniformly quasi Lipschitzian mappings in convex metric spaces is defined. Without using "the rate of convergence condition" $\sum_{n=0}^{\infty}(k_n-1)<\infty$ associated with asymptotically (quasi-)nonexpansive mappings, some convergence theorems are also proved. The results presented generalize, improve and unify some recent results.

KEYWORDS: Uniformly quasi-Lipschitzian mappings; Common fixed points; Convex metric spaces.
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1. INTRODUCTION AND PRELIMINARIES

Throughout this paper, $\mathbb{N}$ denotes the set of natural numbers. We also denote by $F(T)$ the set of fixed points of T and by $F=F(T)\cap F(S)$ the set of common fixed points of two mappings T and S.

Let (X,d) be a metric space. A mapping $T:X\to X$ is said to be asymptotically nonexpansive, if there exists a sequence $k_n\in[1,\infty)$ with $\lim_{n\to\infty}k_n=1$ such that

$$d(T^nx,T^ny)\le k_nd(x,y),\ \ \forall x,y\in X,\ n\in\mathbb{N}.$$

If $F(T)\neq\emptyset$, then T is said to be asymptotically quasi-nonexpansive, if there exists $k_n\in[1,\infty)$ with $\lim_{n\to\infty}k_n=1$ such that

$$d(T^nx,p)\le k_nd(x,p),\ \ \forall x\in X,\ p\in F(T),\ n\in\mathbb{N}.$$

T is said to be uniformly quasi-Lipschitzian, if there exists a constant $L>0$ (called Lipschitz constant) such that

$$d(T^nx,p)\le Ld(x,p),\ \ \forall x\in X,\ p\in F(T),\ n\in\mathbb{N}.$$

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Remark 1.1. If $F(T) \neq \emptyset$, it follows from the above definitions that each asymptotically nonexpansive mapping must be an asymptotically quasi-nonexpansive, and each asymptotically quasi-nonexpansive mapping must be a uniformly quasi-Lipschitzian, where $L = \sup_{n\geq 0}\{k_n\} < \infty$. But the converse may not necessarily hold.

The approximation problems concerned with the fixed points of the asymptotically nonexpansive mappings and asymptotically quasi-nonexpansive mappings have been studied extensively by many authors in recent years. Takahashi [4] introduced the notion of a convex metric space and studied the fixed point theory for nonexpansive mappings in such a setting. A normed linear space is a special example of a convex metric space. But there are many examples of convex metric spaces which are not embedded in any normed linear space (see [4]). Later on, Tian [5] gave some sufficient and necessary conditions such that Ishikawa iteration process for an asymptotically quasi-nonexpansive mapping converges to a fixed point in a convex metric space. Liu et al. [3] and Wang and Liu [6] gave some sufficient and necessary conditions for Ishikawa iteration process with errors to approximate common fixed points of two uniformly quasi-Lipschitzian mappings in a convex metric space. Also, Chang et al. [1], Khan and Abbas [2], Yildirim and Khan [8] and other authors have studied fixed point theorems in convex metric spaces.

We recall the following which can be found in [5].

Let (X, d) be a metric space.

- A mapping $W : X^3 \times [0,1]^3 \to X$ is said to be a convex structure on X, if it satisfies the following condition: for any $(x, y, z; a, b, c) \in X^3 \times [0,1]^3$ with $a + b + c = 1$, and $u \in X$:
$$d(W(x, y, z; a, b, c), u) \leq ad(x, u) + bd(y, u) + cd(z, u).$$
- If (X, d) is a metric space with a convex structure W, then (X, d) is called a convex metric space.
- Let (X, d) be a convex metric space, a nonempty subset E of X is said to be convex, if $W(x, y, z; a, b, c) \in E$, $\forall (x, y, z) \in E^3$, $(a, b, c) \in [0,1]^3$ with $a + b + c = 1$.

Recently, Wang and Liu [6] considered the following iteration process for uniformly quasi-Lipschitzian mappings S and T in convex metric spaces:

$$\begin{aligned} x_{n+1} &= W(x_n, S^n y_n, u_n; a_n, b_n, c_n), \\ y_n &= W\left(x_n, T^n x_n, v_n; a_n^{'}, b_n^{'}, c_n^{'}\right) \end{aligned} \tag{1.1}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{a_n'\}, \{b_n'\}, \{c_n'\}$ are six sequences in $[0,1]$ with $a_n + b_n + c_n = a_n^{'} + b_n^{'} + c_n^{'} = 1$, $n \in \mathbb{N}$ and $\{u_n\}, \{v_n\}$ are two sequences in X satisfying condition: For any nonnegative integers $n, m, 0 \leq n < m$, if $\delta(A_{nm}) > 0$, then

$$\max_{n\leq i,j\leq m}\{d(x, y) : x \in \{u_i, v_i\}, y \in \{x_j, y_j, Sy_j, Tx_j, u_j, v_j\}\} < \delta(A_{nm}),$$

where $A_{nm} = \{x_i, y_i, Sy_i, Tx_i, u_i, v_i : n \leq i \leq m\}$,

$$\delta(A_{nm}) = \sup_{x,y\in A_{nm}} d(x, y).$$

They also proved convergence of the iteration process (1.1) to a common fixed point of S and T.

Motivated by the above studies, we introduce, in this paper, an iteration process to approximate common fixed points for two uniformly quasi-Lipschitzian mappings as follows:

Let (X, d) be a convex metric space with a convex structure W. Let $S, T : X \to X$ be uniformly quasi-Lipschitzian mappings with respective Lipschitz constants $L_1 > 0$ and $L_2 > 0, \{a_n\}, \{b_n\}, \{c_n\}$ be three sequences in $[0, 1]$ with $a_n + b_n + c_n = 1$, $n \in \mathbb{N}$. For any given $x_0 \in X$, define a sequence $\{x_n\}$ as follows:

$$x_{n+1} = W\left(x_n, S^n x_n, T^n x_n; a_n, b_n, c_n\right). \tag{1.2}$$

While acknowledging the process (1.1) due to Wang and Liu, we underscore that our process

- is independent of (1.1) due to Wang and Liu : none reduces to the other.
- is one-step process as compared with the two-step process (1.1) and still able to compute common fixed points.
- being one-step process is simpler than (1.1).

Having introduced this process, we use it to prove some strong convergence results for quasi-Lipschitzian mappings. Moreover, as opposed to Wang and Liu [6], some convergence theorems are proved for asymptotically (quasi-)nonexpansive mappings without using "the rate of convergence condition" $\sum_{n=0}^{\infty}(k_n - 1) < \infty$ associated with such mappings.

In order to prove our main results, the following lemma will be needed:

Lemma 1.2. [9] *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of non-negative numbers such that*

$$a_{n+1} \leq (1 + b_n)\, a_n, \quad n \in \mathbb{N}.$$

If $\sum_{n=1}^{\infty} b_n < +\infty$, then $\lim_{n\to\infty} a_n$ exists.

2. MAIN RESULTS

In what follows, we take $L = \max\{L_1, L_2\}$ where $L_1 > 0$ and $L_2 > 0$ are Lipschitz constants of the quasi-Lipschitzian mappings S and T respectively.

Theorem 2.1. *Let (X, d) be a convex metric space, E be a nonempty closed convex subset of X and $S, T : E \to E$ be uniformly quasi-Lipschitzian mappings. Let the sequence $\{x_n\}$ be as in (1.2) with the sequences $\{a_n\}, \{b_n\}$ and $\{c_n\}$ in $[0, 1]$ satisfying*

$$a_n + b_n + c_n = 1 \text{ and } \sum_{n=0}^{\infty}(1 - a_n) < \infty.$$

If $F \neq \emptyset$, then:

(1) *for all $p \in F$ and for each $n \in \mathbb{N}$,*

$$d\left(x_{n+1}, p\right) \leq \left(1 + L\left(1 - a_n\right)\right) d\left(x_n, p\right),$$

(2) *there exists a constant $M > 0$ such that, for all $n, m \in \mathbb{N}$ and for every $p \in F$,*

$$d\left(x_{n+m}, p\right) \leq M d\left(x_n, p\right).$$

Proof. (1) For any $p \in F$, from (1.2), we have

$$\begin{aligned} d(x_{n+1}, p) &= d(W\left(x_n, S^n x_n, T^n x_n; a_n, b_n, c_n\right), p) \\ &\leq a_n d(x_n, p) + b_n d(S^n x_n, p) + c_n d(T^n x_n, p) \\ &\leq a_n d(x_n, p) + b_n L_1 d(x_n, p) + c_n L_2 d(x_n, p) \\ &\leq \left(a_n + b_n L + c_n L\right) d(x_n, p) \end{aligned}$$

$$\leq \quad (1+L(1-a_n))\, d(x_n,p). \tag{2.1}$$

This completes the proof of (1).

(2) It is well known that $1+x \leq e^x$ for all $x \geq 0$. Using it for the inequality (2.1), we have

$$\begin{aligned} d(x_{n+m},p) &\leq (1+L(1-a_{n+m-1}))\, d(x_{n+m-1},p) \\ &\leq e^{L(1-a_{n+m-1})} d(x_{n+m-1},p) \\ &\leq e^{L(1-a_{n+m-1})} \left[(1+L(1-a_{n+m-2}))\, d(x_{n+m-2},p)\right] \\ &\leq e^{L[(1-a_{n+m-1})+(1-a_{n+m-2})]} d(x_{n+m-2},p) \\ &\ \ \vdots \\ &\leq M d(x_n,p), \end{aligned} \tag{2.2}$$

where $M = e^{L\sum_{k=0}^{\infty}(1-a_k)}$. This completes the proof of (2). □

Now we give the main theorems of this paper. Our first theorem deals with uniformly quasi-Lipschitzian mappings.

Theorem 2.2. *Let (X,d) be a complete convex metric space, E be a nonempty closed convex subset of X and $S,T : E \to E$ be uniformly quasi-Lipschitzian mappings and $F \neq \emptyset$. Suppose that $\{x_n\}$ is the iteration process defined by (1.2), and $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are three sequences in $[0,1]$ satisfying*

$$a_n + b_n + c_n = 1 \text{ and } \sum_{n=0}^{\infty}(1-a_n) < \infty.$$

Then $\{x_n\}$ converges to a fixed point of S and T if and only if $\liminf_{n\to\infty} d(x_n,F) = 0$, where $d(x,F) = \inf\{d(x,p) : p \in F\}$.

Proof. The necessity is obvious. Thus, we will only prove the sufficiency. From Theorem 2.1, we have

$$d(x_{n+1},F) \leq (1+L(1-a_n))\, d(x_n,F).$$

As $\sum_{n=0}^{\infty}(1-a_n) < \infty$, therefore $\lim_{n\to\infty} d(x_n,F)$ exists by Lemma 1.2. But by hypothesis, $\liminf_{n\to\infty} d(x_n,F) = 0$, therefore we must have $\lim_{n\to\infty} d(x_n,F) = 0$.

Next we show that $\{x_n\}$ is a Cauchy sequence. Since $\lim_{n\to\infty} d(x_n,F) = 0$, so for each $\varepsilon > 0$ there exists $n_1 \in \mathbb{N}$ such that

$$d(x_n,F) < \frac{\varepsilon}{M+1} \quad \forall n \geq n_1. \tag{2.3}$$

Thus, there exists $p_1 \in F$ such that

$$d(x_n,p_1) < \frac{\varepsilon}{M+1} \quad \forall n \geq n_1. \tag{2.4}$$

From (2.2) and (2.4), we obtain

$$\begin{aligned} d(x_{n+m},x_n) &\leq d(x_{n+m},p_1) + d(x_n,p_1) \\ &\leq M d(x_n,p_1) + d(x_n,p_1) \\ &= (M+1)\, d(x_n,p_1) \\ &< (M+1)\left(\frac{\varepsilon}{M+1}\right) \\ &= \varepsilon, \end{aligned}$$

for all $n, m \geq n_1$. Hence $\{x_n\}$ is a Cauchy sequence in closed convex subset E of the complete metric space X, therefore, it must converge to a point of E. Suppose $\lim_{n\to\infty} x_n = p$; we prove that $p \in F$. To this end, we only need to prove that F is closed because

$$d(p, F) = \lim_{n\to\infty} d(x_n, F) = 0. \tag{2.5}$$

Let $p_n \in F$ be a sequence such that $\lim_{n\to\infty} p_n = p^*$. We show that $p^* \in F$. In fact,

$$\begin{aligned} d(Sp^*, p^*) &\leq d(p^*, p_n) + d(Sp^*, p_n) \\ &\leq d(p^*, p_n) + Ld(p^*, p_n) \\ &= (1+L)\, d(p^*, p_n) \end{aligned}$$

yields that $d(Sp^*, p^*) = 0$. Similarly, $d(Tp^*, p^*) = 0$. Thus $p^* \in F$ and so F is closed. Thus by (2.5), $p \in F$. This completes the proof. □

In the following results concerned with asymptotically (quasi-)nonexpansive mappings, we do not need "the rate of convergence condition" $\sum_{n=0}^{\infty}(k_n - 1) < \infty$ associated with such type of mappings.

Theorem 2.3. *Let (X, d) be a complete convex metric space, E be a nonempty closed convex subset of X and $S, T : E \to E$ be asymptotically quasi-nonexpansive mappings with sequences $\{k_n\}$ and $\left\{k_n^{'}\right\}$ (without the conditions $\sum_{n=0}^{\infty}(k_n - 1) < \infty$ and $\sum_{n=0}^{\infty}\left(k_n^{'} - 1\right) < \infty$), and $F \neq \emptyset$. Suppose that $\{x_n\}$ is the iteration process defined by (1.2), and $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are three sequences in $[0, 1]$ satisfying*

$$a_n + b_n + c_n = 1 \text{ and } \sum_{n=0}^{\infty}(1 - a_n) < \infty.$$

Then $\{x_n\}$ converges to a fixed point of S and T if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.

Proof. $\{k_n\}, \left\{k_n^{'}\right\} \subset [1, \infty)$ and $\lim_{n\to\infty} k_n = \lim_{n\to\infty} k_n^{'} = 1$; therefore there exist $L_1 > 0$ and $L_2 > 0$ such that $L_1 = \sup_{n\geq 0}\{k_n\} < \infty$ and $L_2 = \sup_{n\geq 0}\left\{k_n^{'}\right\} < \infty$. In this case, S and T are uniformly quasi-Lipschitzian mappings with $L_1 > 0$ and $L_2 > 0$. Hence, Theorem 2.3 can be proven by Theorem 2.2. □

Theorem 2.4. *Let (X, d) be a complete convex metric space, E be a nonempty closed convex subset of X and $S, T : E \to E$ be asymptotically nonexpansive mappings with sequences $\{k_n\}$ and $\left\{k_n^{'}\right\}$ (without the conditions $\sum_{n=0}^{\infty}(k_n - 1) < \infty$ and $\sum_{n=0}^{\infty}\left(k_n^{'} - 1\right) < \infty$), and $F \neq \emptyset$. Suppose that $\{x_n\}$ is the iteration process defined by (1.2), and $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are three sequences in $[0, 1]$ satisfying*

$$a_n + b_n + c_n = 1 \text{ and } \sum_{n=0}^{\infty}(1 - a_n) < \infty.$$

Then $\{x_n\}$ converges to a fixed point of S and T if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.

Remark 2.1. All the results proved in this paper can also be proved for the iteration process with error terms. In this case our main iteration process (1.2) looks like

$$x_{n+1} = W(x_n, S^n x_n, T^n x_n, u_n; a_n, b_n, c_n, d_n), \tag{2.6}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}$ are sequences in $[0,1]$ with $a_n + b_n + c_n + d_n = 1$, $n \in \mathbb{N}$.

Remark 2.2. (i) From computational point of view, our iteration processes (1.2) and (2.6) are simpler than iteration processes of Chang et al. [1], Liu et al. [3], Wang and Liu [6].

(ii) Our results also generalize results of Yao et al. [7] to two uniformly quasi-Lipschitzian mappings in convex metric spaces.

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