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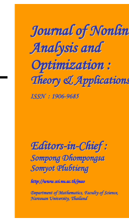

NECESSARY AND SUFFICIENT OPTIMALITY CONDITIONS FOR FVPS WITH GENERALIZED BOUNDARY CONDITIONS

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ABSTRACT. This paper presents the necessary and sufficient optimality conditions for fractional variational problems with generalized boundary conditions. The Lagrangian that we consider depends on left Caputo fractional derivatives of order α and which also depends on the boundary points $y(a)$ and $y(b)$,the terminal time b, and the end point $y(b)$ may be not specified . Examples are presented to demonstrate the application of the formulations.

KEYWORDS : Necessary and sufficient optimality conditions; Caputo fractional derivatives; Generalized boundary conditions; Fractional variational problems.

1. INTRODUCTION

Fractional calculus is the branch of mathematics that generalizes the derivative and the integral of a function to a noninteger order. The study of fractional problems of the calculus of variations is a subject of current strong research due to its many applications in science, engineering, mechanics, chemistry, biology, economics and control theory (see [3, 4, 8–10, 13]).

The fractional calculus of variations was introduced by Riewe in [11, 12], where he developed Hamiltonian, and other concepts of classical mechanics using fractional calculus.

Klimek presented a fractional sequential mechanics model with symmetric fractional derivatives [5] and stationary conservation laws for fractional differential equations with variable coefficients [6].

In [1], Agrawal combined the calculus of variations and the concepts of fractional derivatives to obtain the fractional variational problems.

In [2], Agrawal considered the functional which had unspecified end points. The extremal function satisfied the terminal condition $y(a) = y_a$ and intersected the curve $z = c(x)$ for the first time at b , i.e. $y(b) = c(b)$. Here c(x) was the specified curve.

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In [7], the two authors discussed the necessary optimality conditions for the functionals of the form: $J(y) = I^{\gamma}_{a+}L(x, y(x), {}^{R}D^{\alpha}_{a+}y, y(a))$.

In this paper, we will develop the theory of fractional variational calculus further by proving the necessary and sufficient optimality conditions for more general problems. The functional that we consider has the form:

$$J(y) = \int_a^b L(x, y(x), {}^{C}D^{\alpha}_{a+}y(x), y(a), y(b))dx \longrightarrow min \qquad \text{(VP)}$$

where $L(x, y, u, v, w)$ is a function with continuous first and second (partial) derivatives with respect to all its arguments. And $y(x)$ has continuous left Caputo fractional derivatives of order α, here $0 < \alpha < 1$. The initial time $x = a$ is specified, while the initial point $y(a)$, the terminal time b , and the end point $y(b)$ may be not specified. In some cases, the end point $y(x)$ may intersect a specified curve at the terminal time.

2. PRELIMINARIES

2.1. **REVIEW ON FRACTIONAL CALCULUS.** Let $f : [a, b] \longrightarrow$ R be a function, α a positive real number, n the integer satisfying that $n - 1 \leq \alpha < n$,and Γ the Euler gamma function.

2.1.1.The left and right Riemann-Liouville fractional integrals of order α are defined by

$$I^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt$$

and

$$I^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt$$

respectively;

2.1.2. The left and right Riemann-Liouville fractional derivatives of order α are defined by

$$D^{\alpha}_{a+}f(x) = \frac{d^n}{dx^n} I^{n-\alpha}_{a+} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-t)^{n-\alpha-1} f(t)dt$$

and

$$D^{\alpha}_{b-}f(x) = (-1)^n \frac{d^n}{dx^n} I^{n-\alpha}_{b-} f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_x^b (t-x)^{n-\alpha-1} f(t)dt$$

respectively;

2.1.3. The left and right Caputo fractional derivatives of order α are defined by

$${}^{C}D^{\alpha}_{a+}f(x) = I^{n-\alpha}_{a+} \frac{d^n}{dx^n} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^{(n)}(t)dt$$

and

$${}^{C}D^{\alpha}_{b-}f(x) = (-1)^n I^{n-\alpha}_{b-} \frac{d^n}{dx^n} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_x^b (-1)^n (t-x)^{n-\alpha-1} f^{(n)}(t)dt$$

respectively.

2.2. **INTEGRATION BY PARTS FOR CAPUTO FRACTIONAL DERIVATIVES.**

$$\int_a^b g(x) \cdot^C D_{a+}^{\alpha} f(x)dx = \int_a^b f(x) \cdot D_{b-}^{\alpha} g(x)dx + \sum_{j=0}^{n-1}[D_{b-}^{\alpha+j-n} g(x) \cdot D_{b-}^{n-1-j} f(x)]_a^b$$

and

$$\int_a^b g(x) \cdot^C D_{b-}^{\alpha} f(x)dx = \int_a^b f(x) \cdot D_{a+}^{\alpha} g(x)dx + \sum_{j=0}^{n-1}[(-1)^{n+j} D_{a+}^{\alpha+j-n} g(x) \cdot D_{a+}^{n-1-j} f(x)]_a^b$$

Therefore, if $0 < \alpha < 1$, we obtain

$$\int_a^b g(x) \cdot^C D_{a+}^{\alpha} f(x)dx = \int_a^b f(x) \cdot D_{b-}^{\alpha} g(x)dx + [I_{b-}^{1-\alpha} g(x) \cdot f(x)]_a^b$$

and

$$\int_a^b g(x) \cdot^C D_{b-}^{\alpha} f(x)dx = \int_a^b f(x) \cdot D_{a+}^{\alpha} g(x)dx - [I_{a+}^{1-\alpha} g(x) \cdot f(x)]_a^b$$

Moreover, if f is a function such that $f(a) = f(b) = 0$,we have the simpler formulas:

$$\int_a^b g(x) \cdot^C D_{a+}^{\alpha} f(x)dx = \int_a^b f(x) \cdot D_{b-}^{\alpha} g(x)dx$$

and

$$\int_a^b g(x) \cdot^C D_{b-}^{\alpha} f(x)dx = \int_a^b f(x) \cdot D_{a+}^{\alpha} g(x)dx$$

3. Necessary optimality conditions

We will discuss the necessary optimality conditions for the problem (VP) in different cases of boundary conditions.

Case1. The initial point y(a) is specified .

Theorem 3.1. *If the terminal time $x = b$ and the end point y(b) are specified as well. Assume that $Y(x)$ is the desired function, which satisfies that $Y(a) = Y_a$ and $Y(b) = Y_b$. Then, $Y(x)$ satisfies the fractional E-L equation:*

$$\frac{\partial L}{\partial Y} + D_{b-}^{\alpha} \frac{\partial L}{\partial^C D_{a+}^{\alpha} Y} = 0$$

Proof. Define a family of curves:

$$y(x) = Y(x) + \varepsilon\eta(x) \tag{3.1}$$

where $\eta(x)$ is a variation of $y(x)$.
Substituting (3.1) into (VP), we obtain

$$J(\varepsilon) = \int_a^b L(x, Y + \varepsilon\eta,^C D_{a+}^{\alpha}(Y + \varepsilon\eta), Y(a) + \varepsilon\eta(a), Y(b) + \varepsilon\eta(b))dx$$

For simplicity, we write $L(x, Y + \varepsilon\eta,^C D_{a+}^{\alpha}(Y + \varepsilon\eta), Y(a) + \varepsilon\eta(a), Y(b) + \varepsilon\eta(b))$ as $\widehat{L}$,
and write $L(x, Y,^C D_{a+}^{\alpha} Y, Y(a), Y(b))$ as L,
Find the expression for $dJ/d\varepsilon$

$$\frac{dJ}{d\varepsilon} = \int_a^b \partial_2\widehat{L} \cdot \eta + \partial_3\widehat{L} \cdot^C D_{a+}^{\alpha}\eta + \partial_4\widehat{L} \cdot \eta(a) + \partial_5\widehat{L} \cdot \eta(b)dx$$

Notice that $J(\varepsilon) \geq J(0)$,then

$$\frac{dJ}{d\varepsilon}\mid_{\varepsilon=0}= \int_a^b \partial_2 L\cdot\eta + \partial_3 L^C D^\alpha_{a+}\eta + \partial_4 L\cdot\eta(a) + \partial_5 L\cdot\eta(b)dx = 0$$

Using integration by parts

$$\begin{aligned}&\int_a^b \eta(\partial_2 L + D^\alpha_{b-}\partial_3 L)dx + \eta(I^{1-\alpha}_{b-}\partial_3 L)\mid_a^b + \int_a^b \eta(a)\partial_4 L + \eta(b)\partial_5 L dx\\ &= \int_a^b \eta(\frac{\partial L}{\partial Y} + D^\alpha_{b-}\frac{\partial L}{\partial^C D^\alpha_{a+}Y})dx + \eta(I^{1-\alpha}_{b-}\frac{\partial L}{\partial^C D^\alpha_{a+}Y})\mid_a^b\\ &+ \int_a^b \eta(a)\frac{\partial L}{\partial Y(a)}dx + \int_a^b \eta(b)\frac{\partial L}{\partial Y(b)}dx = 0\end{aligned} \tag{3.2}$$

Since $\eta(x)$ is arbitrary, $I^{1-\alpha}_{b-}\frac{\partial L}{\partial^C D^\alpha_{a+}Y})\mid_{x=b}= 0$, $\eta(a) = 0$, and $\eta(b) = 0$, which gives the fractional E-L equation in the form

$$\frac{\partial L}{\partial Y} + D^\alpha_{b-}\frac{\partial L}{\partial^C D^\alpha_{a+}Y} = 0 \tag{3.3}$$

□

Theorem 3.2. *If the terminal time $x = b$ is specified, while the end point $y(b)$ is unspecified. Assume that $Y(x)$ is the desired function. Then, $Y(x)$ satisfies the following fractional E-L equation and the transversality condition.*

$$\frac{\partial L}{\partial Y} + D^\alpha_{b-}\frac{\partial L}{\partial^C D^\alpha_{a+}Y} = 0$$

and

$$\int_a^b \frac{\partial L}{\partial Y(b)}dx = 0$$

Proof. Using the same way as theorem 3.1, we get

$$\begin{aligned}\frac{dJ}{d\varepsilon}\mid_{\varepsilon=0}&= \int_a^b \eta(\frac{\partial L}{\partial Y} + D^\alpha_{b-}\frac{\partial L}{\partial^C D^\alpha_{a+}Y})dx + \eta(I^{1-\alpha}_{b-}\frac{\partial L}{\partial^C D^\alpha_{a+}Y})\mid_a^b\\ &+ \int_a^b \eta(a)\frac{\partial L}{\partial Y(a)}dx + \int_a^b \eta(b)\frac{\partial L}{\partial Y(b)}dx = 0\end{aligned}$$

Since $\eta(x)$ is arbitrary, which gives the fractional E-L equation as

$$\frac{\partial L}{\partial Y} + D^\alpha_{b-}\frac{\partial L}{\partial^C D^\alpha_{a+}Y} = 0$$

Since $I^{1-\alpha}_{b-}\frac{\partial L}{\partial^C D^\alpha_{a+}Y}\mid_{x=b}= 0$ and $\eta(a) = 0$,we have

$$\int_a^b \eta(b)\frac{\partial L}{\partial Y(b)}dx = \eta(b)\int_a^b \frac{\partial L}{\partial Y(b)}dx = 0$$

We know that $\eta(b)$ is arbitrary,which gives the transversality condition as

$$\int_a^b \frac{\partial L}{\partial Y(b)}dx = 0 \tag{3.4}$$

□

Theorem 3.3. *If the terminal time $x = b$ and the end point $y(b)$ are both unspecified, and the function $y(x)$ intersects the curve $z = c(x)$ for the first time at $x = b$, i.e. $y(b) = c(b)$. Here $z = c(x)$ is the specified curve.*

Assume that $Y(x)$ is the desired function, which intersects the curve $z = c(x)$ at $x = B$, i.e. $Y(B) = c(B)$. Then, $Y(x)$ satisfies the following fractional E-L equation and the transversality condition.

$$\frac{\partial L}{\partial Y} + D^{\alpha}_{b-}\frac{\partial L}{\partial^C D^{\alpha}_{a+}Y} = 0$$

and

$$[(Dc - DY)\int_a^B \frac{\partial L}{\partial Y(B)}dx + L] \mid_{x=B} = 0$$

Proof. Define a family of curves

$$y(x) = Y(x) + \varepsilon\eta(x) \tag{3.5}$$

We further define a set of end points:

$$b = B + \varepsilon\delta(b) \tag{3.6}$$

Where $\delta(b)$ is a variation of b.
Substituting (3.5) and (3.6) into (VP), then the equation (VP) can be rewritten as

$$J(\varepsilon) = \int_a^{B+\varepsilon\delta(b)} L[x, Y + \varepsilon\eta, ^C D^{\alpha}_{a+}(Y + \varepsilon\eta), Y(a) + \varepsilon\eta(a), Y(B + \varepsilon\delta(b)) + \varepsilon\eta(B + \varepsilon\delta(b))]dx$$

For simplicity, we write $L[x, Y+\varepsilon\eta, ^C D^{\alpha}_{a+}(Y+\varepsilon\eta), Y(a)+\varepsilon\eta(a), Y(B+\varepsilon\delta(b))+\varepsilon\eta(B+\varepsilon\delta(b))]$ as $\widehat{L}$, and write $L[x, Y, ^C D^{\alpha}_{a+}Y, Y(a), Y(B)]$ as L. Then

$$\begin{aligned}\frac{dJ}{d\varepsilon} = &\int_a^{B+\varepsilon\delta(b)} \partial_2\widehat{L}\cdot\eta + \partial_3\widehat{L}\cdot ^C D^{\alpha}_{a+}\eta + \partial_4\widehat{L}\cdot\eta(a) + \partial_5\widehat{L}\cdot\eta(B + \varepsilon\delta(b))dx\\ &+\delta(B + \varepsilon\delta(b))L[B + \varepsilon\delta(b), Y + \varepsilon\eta, ^C D^{\alpha}_{a+}(Y + \varepsilon\eta),\\ &Y(a) + \varepsilon\eta(a), Y(B + \varepsilon\delta(b)) + \varepsilon\eta(B + \varepsilon\delta(b))]\end{aligned}$$

Notice that $J(\varepsilon) \geq J(0)$, then we have

$$\begin{aligned}\frac{dJ}{d\varepsilon}\mid_{\varepsilon=0} = &\int_a^B \partial_2 L\cdot\eta + \partial_3 L\cdot ^C D^{\alpha}_{a+}\eta + \partial_4 L\cdot\eta(a) + \partial_5 L\cdot\eta(B)dx\\ &+L(B, Y, ^C D^{\alpha}_{a+}Y, Y(a), Y(B))\delta(B)\end{aligned}$$

Using integration by parts

$$\begin{aligned}\frac{dJ}{d\varepsilon}\mid_{\varepsilon=0} = &\int_a^B \eta(\partial_2 L + D^{\alpha}_{B-}\partial_3 L)dx + \eta\cdot(I^{1-\alpha}_{B-}\partial_3 L)\mid_a^B + \int_a^B \partial_4 L\cdot\eta(a)dx\\ &+\int_a^B \partial_5 L\cdot\eta(B)dx + L(B, Y, ^C D^{\alpha}_{a+}Y, Y(a), Y(B))\delta(B) = 0\end{aligned}$$

Since $\eta(x)$ is arbitrary, which gives the fractional E-L equation as

$$\frac{\partial L}{\partial Y} + D^{\alpha}_{B-}\frac{\partial L}{\partial^C D^{\alpha}_{a+}Y} = 0$$

Since $I^{1-\alpha}_{B-}\partial_3 L\mid_{x=B} = 0$ and $\eta(a) = 0$, we obtain

$$\eta(B)\int_a^B \frac{\partial L}{\partial Y(B)}dx + \delta(B)L(B, Y, ^C D^{\alpha}_{a+}Y, Y(a), Y(B)) = 0 \tag{3.7}$$

We notice that

$$y(b) = c(b) \tag{3.8}$$

By using (3.5) and (3.6), the equation (3.8) becomes

$$Y(B+\varepsilon\delta(b))+\varepsilon\eta(B+\varepsilon\delta(b))=c(B+\varepsilon\delta(b)) \tag{3.9}$$

Differentiating equation (3.9) with respect to ε and then setting $\varepsilon=0$,we get

$$DY(B)\cdot\delta(B)+\eta(B)=Dc(B)\cdot\delta(B) \tag{3.10}$$

Here $D(\cdot)=\frac{d(\cdot)}{dx}$
We further get

$$\eta(B)=\delta(B)D(c(B)-Y(B)) \tag{3.11}$$

Substituting (3.11) into (3.7), then (3.7) becomes the form

$$\delta(B)[(Dc-DY)\int_a^B\frac{\partial L}{\partial Y(B)}dx+L]\mid_{x=B}=0$$

Since $\delta(B)$ is arbitrary, then the transversality condition is given below

$$[(Dc-DY)\int_a^B\frac{\partial L}{\partial Y(B)}dx+L]\mid_{x=B}=0 \tag{3.12}$$

□

Case 2. The initial point $y(a)$ is unspecified.

Theorem 3.4. *If the terminal time $x=b$ and the end point $y(b)$ are both specified. Assume that $Y(x)$ is the desired function, which satisfies that $Y(b)=Y_b$. Then, $Y(x)$ satisfies the following fractional E-L equation and the transversality condition.*

$$\frac{\partial L}{\partial Y}+D_{b-}^{\alpha}\frac{\partial L}{\partial^C D_{a+}^{\alpha}Y}=0$$

and

$$\int_a^b\frac{\partial L}{\partial Y(a)}dx-\frac{1}{\Gamma(1-\alpha)}\int_a^b(t-a)^{-\alpha}\frac{\partial L(t)}{\partial^C D_{a+}^{\alpha}Y(t)}dt=0$$

Proof. Using the same way as theorem 3.1, we get

$$\frac{dJ}{d\varepsilon}\mid_{\varepsilon=0}=\int_a^b\eta(\frac{\partial L}{\partial Y}+D_{b-}^{\alpha}\frac{\partial L}{\partial^C D_{a+}^{\alpha}Y})dx+\eta(I_{b-}^{1-\alpha}\frac{\partial L}{\partial^C D_{a+}^{\alpha}Y})\mid_a^b$$
$$+\int_a^b\eta(a)\frac{\partial L}{\partial Y(a)}dx+\int_a^b\eta(b)\frac{\partial L}{\partial Y(b)}dx=0$$

Since $\eta(x)$ is arbitrary, which gives the fractional E-L equation as

$$\frac{\partial L}{\partial Y}+D_{b-}^{\alpha}\frac{\partial L}{\partial^C D_{a+}^{\alpha}Y}=0$$

Since $\eta(b)=0$ and $I_{b-}^{1-\alpha}\frac{\partial L}{\partial^C D_{a+}^{\alpha}Y}\mid_{x=b}=0$,we have

$$\eta(a)[\int_a^b\frac{\partial L}{\partial Y(a)}dx-I_{b-}^{1-\alpha}\frac{\partial L}{\partial^C D_{a+}^{\alpha}Y}\mid_{x=a}]=0 \tag{3.13}$$

We know that $\eta(a)$ is arbitrary, therefore (3.13) becomes

$$\int_a^b\frac{\partial L}{\partial Y(a)}dx-I_{b-}^{1-\alpha}\frac{\partial L}{\partial^C D_{a+}^{\alpha}Y}\mid_{x=a}=0 \tag{3.14}$$

We further get the transversality conditions as follow

$$\int_a^b \frac{\partial L}{\partial Y(a)}dx - \frac{1}{\Gamma(1-\alpha)}\int_a^b (t-a)^{-\alpha}\frac{\partial L(t)}{\partial^C D_{a+}^\alpha Y(t)}dt = 0$$

□

Theorem 3.5. *If the terminal time $x = b$ is specified, while the end point $y(b)$ is unspecified. Assume that $Y(x)$ is the desired function. Then, $Y(x)$ satisfies the following fractional E-L equation and the transversality condition.*

$$\frac{\partial L}{\partial Y} + D_{b-}^\alpha \frac{\partial L}{\partial^C D_{a+}^\alpha Y} = 0$$

and

$$\int_a^b \frac{\partial L}{\partial Y(a)}dx - \frac{1}{\Gamma(1-\alpha)}\int_a^b (t-a)^{-\alpha}\frac{\partial L(t)}{\partial^C D_{a+}^\alpha Y(t)}dt = 0$$

$$\int_a^b \frac{\partial L}{\partial Y(b)}dx = 0$$

Proof. Using the same way as theorem 3.1, we get

$$\frac{dJ}{d\varepsilon}\mid_{\varepsilon=0} = \int_a^b \eta\left(\frac{\partial L}{\partial Y} + D_{b-}^\alpha\frac{\partial L}{\partial^C D_{a+}^\alpha Y}\right)dx + \eta\left(I_{b-}^{1-\alpha}\frac{\partial L}{\partial^C D_{a+}^\alpha Y}\right)\mid_a^b$$
$$+\int_a^b \eta(a)\frac{\partial L}{\partial Y(a)}dx + \int_a^b \eta(b)\frac{\partial L}{\partial Y(b)}dx = 0$$

Since $\eta(x)$ is arbitrary,which gives the fractional E-L equation as

$$\frac{\partial L}{\partial Y} + D_{b-}^\alpha \frac{\partial L}{\partial^C D_{a+}^\alpha Y} = 0$$

Since $I_{b-}^{1-\alpha}\frac{\partial L}{\partial^C D_{a+}^\alpha Y}\mid_{x=b}= 0$,we have

$$\eta(a)\left[\int_a^b \frac{\partial L}{\partial Y(a)}dx - I_{b-}^{1-\alpha}\frac{\partial L}{\partial^C D_{a+}^\alpha Y}\mid_{x=a}\right] + \eta(b)\int_a^b \frac{\partial L}{\partial Y(b)}dx = 0$$

We know that $\eta(a)$ and $\eta(b)$ are both arbitrary, the transversality conditions are given below

$$\int_a^b \frac{\partial L}{\partial Y(a)}dx - \frac{1}{\Gamma(1-\alpha)}\int_a^b (t-a)^{-\alpha}\frac{\partial L(t)}{\partial^C D_{a+}^\alpha Y(t)}dt = 0$$

$$\int_a^b \frac{\partial L}{\partial Y(b)}dx = 0$$

□

Theorem 3.6. *If the terminal time $x = b$ and the end point $y(b)$ are both unspecified, and the function $y(x)$ intersects the curve $z = c(x)$ for the first time at $x = b$, i.e. $y(b) = c(b)$, where $z = c(x)$ is the specified curve.*

Assume that $Y(x)$ is the desired function, which intersects the curve $z = c(x)$ at $x = B$, i.e. $Y(B) = c(B)$. Then, $Y(x)$ satisfies the following fractional E-L equation and the transversality condition.

$$\frac{\partial L}{\partial Y} + D_{b-}^\alpha \frac{\partial L}{\partial^C D_{a+}^\alpha Y} = 0$$

and

$$[(Dc - DY)\int_a^B \frac{\partial L}{\partial Y(B)}dx + L]\mid_{x=B}= 0$$

$$\int_a^B \frac{\partial L}{\partial Y(a)}dx - \frac{1}{\Gamma(1-\alpha)}\int_a^B (t-a)^{-\alpha}\frac{\partial L(t)}{\partial^C D^{\alpha}_{a+}Y(t)}dt = 0$$

Proof. Using the same way as theorem 3.1, we obtain

$$\frac{dJ}{d\varepsilon}\mid_{\varepsilon=0}= \int_a^B \eta(\partial_2 L + D^{\alpha}_{B-}\partial_3 L)dx + \eta\cdot(I^{1-\alpha}_{B-}\partial_3 L)\mid_a^B + \int_a^B \partial_4 L\cdot\eta(a)dx$$

$$+\int_a^B \partial_5 L\cdot\eta(B)dx + L(B, Y, ^C D^{\alpha}_{a+}Y, Y(a), Y(B))\delta(B) = 0$$

Since $\eta(x)$ is arbitrary,which gives the fractional E-L equation as

$$\frac{\partial L}{\partial Y} + D^{\alpha}_{B-}\frac{\partial L}{\partial^C D^{\alpha}_{a+}Y} = 0$$

Since $I^{1-\alpha}_{B-}\partial_3 L\mid_{x=B}= 0$,we get

$$\eta(a)(\int_a^B \frac{\partial L}{\partial Y(a)}dx - I^{1-\alpha}_{B-}\frac{\partial L}{\partial^C D^{\alpha}_{a+}Y}\mid_{x=a}) + \eta(B)\int_a^B \frac{\partial L}{\partial Y(B)}dx$$

$$+\delta(B)L(B, Y, ^C D^{\alpha}_{a+}Y, Y(a), Y(B)) = 0 \tag{3.15}$$

From theorem 3.3, we know that

$$\eta(B) = \delta(B)D(c(B) - Y(B)) \tag{3.16}$$

Substituting (3.16) into (3.15), then (3.15) becomes the following form

$$\eta(a)(\int_a^B \frac{\partial L}{\partial Y(a)}dx - I^{1-\alpha}_{B-}\frac{\partial L}{\partial^C D^{\alpha}_{a+}Y}\mid_{x=a}) + \delta(B)[(Dc - DY)\int_a^B \frac{\partial L}{\partial Y(B)}dx + L]\mid_{x=B}= 0$$

Since $\eta(a)$ and $\eta(B)$ are both arbitrary, the transversality conditions are given below

$$[(Dc - DY)\int_a^B \frac{\partial L}{\partial Y(B)}dx + L]\mid_{x=B}= 0$$

and

$$\int_a^B \frac{\partial L}{\partial Y(a)}dx - \frac{1}{\Gamma(1-\alpha)}\int_a^B (t-a)^{-\alpha}\frac{\partial L(t)}{\partial^C D^{\alpha}_{a+}Y(t)}dt = 0$$

□

4. SUFFICIENT CONDITIONS

In this section, we prove the sufficient optimality conditions of the following functional

$$J(y) = \int_a^b L(x, y(x), ^C D^{\alpha}_{a+}y(x), y(a), y(b))dx \longrightarrow min \tag{4.0}$$

Where $0 < \alpha < 1$,the initial time a and the terminal time b are both specified, while, the boundary points $y(a)$ and $y(b)$ are both free.

Some conditions of convexity are in order. Given a function $L = L(x, y, u, v, w)$,we say that L is jointly convex in (y, u, v, w), if $\frac{\partial L}{\partial y}, \frac{\partial L}{\partial u}, \frac{\partial L}{\partial v}, \frac{\partial L}{\partial w}$ exist and are continuous and verify the following condition:

$$L(x,y+y_1,u+u_1,v+v_1,w+w_1)-L(x,y,u,v,w)$$

$$\geq \frac{\partial L}{\partial y}\cdot y_1+\frac{\partial L}{\partial u}\cdot u_1+\frac{\partial L}{\partial v}\cdot v_1+\frac{\partial L}{\partial w}\cdot w_1$$

for all $(x,y,u,v,w),(x,y+y_1,u+u_1,v+v_1,w+w_1)\in[a,b]\times R^4$

Theorem 4.1. *Let $L(x,y,u,v,w)$ be jointly convex in (y,u,v,w). If y satisfies the fractional E-L equation*

$$\frac{\partial L}{\partial Y}+D^{\alpha}_{b-}\frac{\partial L}{\partial^C D^{\alpha}_{a+}Y}=0 \tag{4.1}$$

and the transversality conditions

$$\int_a^b \frac{\partial L}{\partial Y(a)}dx-\frac{1}{\Gamma(1-\alpha)}\int_a^b (t-a)^{-\alpha}\frac{\partial L(t)}{\partial^C D^{\alpha}_{a+}Y(t)}dt=0 \tag{4.2}$$

$$\int_a^b \frac{\partial L}{\partial Y(b)}dx=0 \tag{4.3}$$

Then, y is a global minimizer to the functional (4.0).

Proof. Since L is jointly convex in (y,u,v,w), for any admissable function $y+\eta$, we have

$$J(y+\eta)-J(y)=\int_a^b L(x,y+\eta,^C D^{\alpha}_{a+}(y+\eta),y(a)+\eta(a),y(b)+\eta(b))$$

$$-L(x,y,^C D^{\alpha}_{a+}y,y(a),y(b))dx$$

$$\geq \int_a^b \frac{\partial L}{\partial y}\cdot\eta+\frac{\partial L}{\partial^C D^{\alpha}_{a+}y}{}^C D^{\alpha}_{a+}\eta+\frac{\partial L}{\partial y(a)}\cdot\eta(a)+\frac{\partial L}{\partial y(b)}\cdot\eta(b)dx$$

Using integration by parts

$$J(y+\eta)-J(y)\geq\int_a^b \eta(\frac{\partial L}{\partial y}+D^{\alpha}_{b-}(\frac{\partial L}{\partial^C D^{\alpha}_{a+}y}))dx+\eta(a)[\int_a^b \frac{\partial L}{\partial y(a)}$$

$$-I^{1-\alpha}_{b-}\frac{\partial L}{\partial^C D^{\alpha}_{a+}y}\,|_{x=a}]+\eta(b)\int_a^b\frac{\partial L}{\partial y(b)}dx=0$$

Since y satisfies (4.1)-(4.3), thus we obtain $J(y+\eta)-J(y)\geq 0$.
We can similarly prove the sufficient optimality conditions of the functional (VP) with other different boundary conditions. □

5. ILLUSTRATIVE EXAMPLES

Consider the following functional:

$$J(y)=\int_a^b y^2(x)+(^C D^{\alpha}_{a+}y(x))^2+y(a)^2+y(b)^2dx\longrightarrow min$$

Where the initial time $x=a$ is specified. We will discuss its E-L equations and transversality conditions in different cases of boundary conditions.

Case1. The initial point is specified i.e. $y(a)=y_a$

Example 5.1. If the terminal time $x = b$ and the end point $y(b)$ are both specified.

This problem becomes a specified boundary conditions problem. Assume $y(x)$ is the desired function, we get the generalized fractional E-L equation in the following form:

$$y(x) + D^{\alpha}_{b-}(^{C}D^{\alpha}_{a+}y(x)) = 0$$

Example 5.2. If the terminal time $x = b$ is specified, while the end point $y(b)$ is unspecified.We get the generalized fractional E-L equation and the transversality condition, respectively, in the following form:

$$y(x) + D^{\alpha}_{b-}(^{C}D^{\alpha}_{a+}y(x)) = 0$$

and

$$y(b)(b - a) = 0$$

Example 5.3. If the terminal time $x = b$ and the end point $y(b)$ are both unspecified in advance, and the function $y(x)$ intersects the curve $c(x) = x^2$,for the first time at $x = b$,i.e $y(b) = c(b) = b^2$.

Assume that $y(x)$ is the desired function, and it satisfies $y(b) = b^2$.

For this problem, we get the generalized fractional E-L equation and the transversality condition, respectively, in the following form:

$$y(x) + D^{\alpha}_{b-}(^{C}D^{\alpha}_{a+}y(x)) = 0$$

and

$$2(2b - Dy(b))y(b)(b - a) + L(b, y, ^{C}D^{\alpha}_{a+}y, y(a), y(b)) = 0$$

Case2. The initial point is unspecified.

Example 5.4. If the terminal time $x = b$ and the end point $y(b)$ are both specified. We get the generalized fractional E-L equation in the following form:

$$y(x) + D^{\alpha}_{b-}(^{C}D^{\alpha}_{a+}y(x)) = 0$$

and

$$y(x) + D^{\alpha}_{b-}(^{C}D^{\alpha}_{a+}y(x)) = 0$$

$$y(a)(b - a) - \frac{1}{\Gamma(1-\alpha)}\int_a^b (t-a)^{-\alpha} \cdot ^{C}D^{\alpha}_{a+}y(t)dt = 0$$

Example 5.5. If the terminal time $x = b$ is specified, while the end point $y(b)$ is unspecified.We get the generalized fractional E-L equation in the following form:

$$y(x) + D^{\alpha}_{b-}(^{C}D^{\alpha}_{a+}y(x)) = 0$$

and

$$y(b)(b - a) = 0$$

$$y(a)(b - a) - \frac{1}{\Gamma(1-\alpha)}\int_a^b (t-a)^{-\alpha} \cdot ^{C}D^{\alpha}_{a+}y(t)dt = 0$$

Example 5.6. If the terminal time $x = b$ and the end point $y(b)$ are both unspecified in advance, and the function $y(x)$ intersects the curve $c(x) = x^2$, for the first time at $x = b$, i.e $y(b) = c(b) = b^2$.

Assume that $y(x)$ is the desired function, and it satisfies $y(b) = b^2$.

For this problem, we get the generalized fractional E-L equation and the transversality condition, respectively, in the following form:

$$y(x) + D_{b-}^{\alpha}({}^{C}D_{a+}^{\alpha}y(x)) = 0$$

and

$$y(a)(b-a) - \frac{1}{\Gamma(1-\alpha)}\int_a^b (t-a)^{-\alpha} \cdot {}^{C}D_{a+}^{\alpha}y(t)dt = 0$$

$$2(2b - Dy(b))y(b)(b-a) + L(b, y, {}^{C}D_{a+}^{\alpha}y, y(a), y(b)) = 0$$

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