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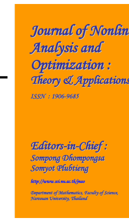

ON THE SEMILOCAL CONVERGENCE OF ULM'S METHOD

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ABSTRACT. We provide sufficient convergence conditions for the semilocal convergence of Ulm's method [9] to a locally unique solution of an equation in a Banach space setting. Our results compare favorably to recent ones by Ezquerro and Hernández [3] which have improved earlier ones [4], [6]–[10], since under the same computational cost we provide: larger convergence domain; finer error bounds on the distances involved, and an at least as precise information on the location of the solution.

KEYWORDS : Ulm's method; Newton's method; Banach space; Recurrence relations; Semilocal convergence; Fréchet derivative.
AMS Subject Classification: 65J15 65H10 65G99 47J25 49M15.

1. INTRODUCTION

In this study we are concerned with the problem of approximating a locally unique solution $x^\star$ of equation

$$F(x) = 0, \tag{1.1}$$

where, F is a Fréchet–differentiable operator defined on a convex subset $\mathcal{D}$ of a Banach space $\mathcal{X}$ with values in a Banach space $\mathcal{Y}$.

A large number of problems in applied mathematics and also in engineering are solved by finding the solutions of certain equations. For example, dynamic systems are mathematically modeled by difference or differential equations and their solutions usually represent the states of the systems. For the sake of simplicity, assume that a time–invariant system is driven by the equation $\dot{x} = Q(x)$, for some suitable operator Q, where x is the state. Then the equilibrium states are determined by solving equation (1.1). Similar equations are used in the case of discrete systems. The unknowns of engineering equations can be functions (difference, differential and integral equations), vectors (systems of linear or nonlinear algebraic

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equations), or real or complex numbers (single algebraic equations with single unknowns). Excpet in special cases, the most commonly used solution methods are iterative–when starting from one or several initial approximations a sequence is constructed that converges to a solution of the equation. Iteration methods are also applied for solving optimization problems. In such cases, the iteration sequences converge to an optimal solution of the problem at hand. Since all of these methods have the same recursive structure, they can be introduced and discussed in a general framework.

In [9], Ulm introduced method

$$\begin{aligned} &B_{n+1} = 2\,B_n - B_n\,F'(x_n)\,B_n \qquad (x_0 \in \mathcal{D}), \quad B_0 \in \mathcal{L}(\mathcal{Y}, \mathcal{X}), \\ &x_{n+1} = x_n - B_n\,F(x_n) \qquad (n \geq 0) \end{aligned} \tag{1.2}$$

to generate a sequence $\{x_n\}$ approximating $x^\star$. Method (1.2) has some usefull properties: First it is like Newton's method, self–correcting. Second, it converges with Newton–like rate. Third, it is inversion free unlike Newton's method. Fourth, apart from solving equation (1.1), the method generates successive approximation: $B_n \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ to the inverse derivative $F'(x^\star)^{-1}$ which important especially when one is interested in solutions sensitive to small perturbations [2], [6].

Ulm [9], Moser [6], Hald [4], Zehnder [10], Petzeltova [7], Potra [8] and others [1], [2] have provided sufficient convergence conditions under various assumptions for the convergence of method (1.2) to $x^\star$.

Recently, Ezquerro and Hernández [3] provided a semilocal convergence analysis for method (1.2) using recurrence relations and conditions which are more general than the mentioned works (see also [5]). They also gave numerical examples where their results hold when the ones by the authors mentioned above do not hold.

Here we are motivated by optimization considerations and the work in [3]. In particular we also provide sufficient convergence conditions for method (1.2) using similar recurrence relations. However under the same computational cost as in [3], our approach has the following advantages:

(a) larger convergence domain;
(b) finer error estimations on the distances $\| x_{n+1} - x_n \|$, $\| x_n - x^\star \|$ ($n \geq 0$);

and

(c) an at least as precise information on the location of the solution of $x^\star$.

2. Semilocal convergence analysis of method (1.2)

To make the paper as self-contained as possible, we re–introduce some of the notations used in [3]. We assume throughout this study:

($\mathcal{H}1$) $\| B_0 \| \leq c_0$,
($\mathcal{H}2$) $\| F(x_0) \| \leq \eta$,
($\mathcal{H}3$) $0 < \| I - F'(x_0)\,B_0 \| \leq a_0 < 1$,
($\mathcal{H}4$) ($\mathcal{H}4$) $\| F'(x) - F'(y) \| \leq \omega(\| x - y \|)$, for all x, $y \in \mathcal{D}$ and some continuous non–decreasing function such that

$$\omega(t\,r) \leq \omega(r)\,t^p \quad \text{for all} \quad r > 0,\ t \in [0,1],\ p \in [0,1].$$

It then follows from ($\mathcal{H}4$) that there exists a continuous and non–decreasing function $\omega_0\ :\ (0, +\infty) \longrightarrow (0, +\infty)$ such that

$$\| F'(x) - F'(x_0) \| \leq \omega_0(\| x - x_0 \|) \quad \text{for all} \quad x \in \mathcal{D}$$

and

$$\omega_0(r) \leq \omega(r) \quad \text{for all} \quad r > 0. \tag{2.1}$$

Clearly $\dfrac{\omega(r)}{\omega_0(r)}$ can be arbitrarily large [1], [2]. Function ω_0 is not used in [3]. It turn out that the introduction of function ω_0 in case it is strictly smaller than ω is the reason for the finer convergence analysis than in [3] that follows:

Let us set

$$b_0 = c_0\,\omega_0(c_0\,\eta). \tag{2.2}$$

Define auxiliary scalar functions g and h by

$$g(x,y) = x + \frac{y}{1+p} \tag{2.3}$$

and

$$h(x,y) = 1 + x + y. \tag{2.4}$$

We shall show that method (1.2) is well defined. Note that if $x_1 \in \mathcal{D}$, then

$$\begin{aligned} \| I - F'(x_1)\,B_0 \| &= \| I - F'(x_0)\,B_0 + (F'(x_0) - F'(x_1))\,B_0 \| \\ &\leq a_0 + \omega_0(\| x_1 - x_0 \|)\ \| B_0 \| = a_0 + b_0; \end{aligned}$$

$$\begin{aligned} \| B_1 - B_0 \| &= \| B_0 - B_0\,F'(x_1)\,B_0 \| \\ &= \| B_0\,(I - F'(x_1)\,B_0) \| \\ &\leq c_0\,(a_0 + b_0); \end{aligned}$$

$$\begin{aligned} \| F(x_1) \| &\leq \| I - F'(x_0)\,B_0 \|\,\| F(x_0) \| + \int_0^1 \omega(t\ \| x_1 - x_0 \|)\,dt\ \| x_1 - x_0 \| \\ &\leq g(a_0,b_0)\ \| F(x_0) \|; \end{aligned}$$

$$\begin{aligned} \| B_1 \| &= \| 2\,B_0 - B_0\,F'(x_1)\,B_0 \| \\ &\leq \| B_0 \| + \| B_0 - B_0\,F'(x_1)\,B_0 \| \\ &\leq c_0 + c_0\,(a_0 + b_0) = c_0\,h(a_0,b_0); \end{aligned}$$

$$\| x_2 - x_1 \| \leq \| B_1 \|\,\| F(x_1) \| \leq g(a_0,b_0)\,h(a_0,b_0)\ \| F(x_0) \|;$$

$$\| x_2 - x_0 \| \leq (1 + g(a_0,b_0)\,h(a_0,b_0))\ \| B_0 \|\,\| F(x_0) \|;$$

and if $x_2 \in \mathcal{D}$, $g(a_0,b_0)\,h(a_0,b_0) < 1$, then we get

$$\| B_1 \|\,\| F'(x_2) - F'(x_1) \| \leq b_0\,g(a_0,b_0)^p\,h(a_0,b_0)^{1+p},$$

and

$$\| I - F'(x_1)\,B_1 \| \leq \| I - F'(x_1)\,B_0 \|^2 \leq (a_0 + b_0)^2,$$

so that

$$\begin{aligned} \| I - F'(x_2)\,B_1 \| &\leq \| I - F'(x_1)\,B_1 \| + \| F'(x_2) - F'(x_1) \|\,\| B_1 \| \\ &\leq (a_0 + b_0)^2 + b_0\,g(a_0,b_0)^p\,h(a_0,b_0)^{1+p}, \end{aligned}$$

and

$$\| B_2 - B_1 \| \leq c_0\,h(a_0,b_0)\left[(a_0 + b_0)^2 + b_0\,g(a_0,b_0)^p\,h(a_0,b_0)^{1+p}\right].$$

Let us set

$$a_1 = (a_0 + b_0)^2, \quad b_1 = b_0\,g(a_0,b_0)^p\,h(a_0,b_0)^{1+p} \quad \text{and} \quad c_1 = c_0\,h(a_0,b_0).$$

Then we can define scalar sequences for all $n \geq 1$:

$$a_n = (a_{n-1} + b_{n-1})^2 \tag{2.5}$$

$$b_n = b_{n-1}\,g(a_{n-1},b_{n-1})^p\,h(a_{n-1},b_{n-1})^{1+p} \tag{2.6}$$

$$c_n = c_{n-1}\,h(a_{n-1},b_{n-1}). \tag{2.7}$$

Let us also define scalar sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ used in [3] as $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ respectively with

$$\gamma_0 = c_0, \quad \alpha_0 = a_0$$

but

$$\beta_0 = c_0\, \omega(c_0\, \eta).$$

Clearly in case function ω_0 is strictly smaller than ω, the our triplet $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ is finer than $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ used in [3].

We shall state the following results but only prove Theorem 2.6, since the rest of the proofs are similar to the corresponding ones in [3] (simply replace the triplet $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ by $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ respectively in the proofs given in [3]):

Lemma 2.1. *Let g and h be the scalar functions given by (2.3) and (2.4) respectively. If a_0 and b_0 satisfy*

$$g(a_0, b_0)^p\, h(a_0, b_0)^{1+p} < 1 \quad \text{and} \quad (a_0 + b_0)^2 < a_0, \tag{2.8}$$

then the following hold true:

(a) *($g(a_0, b_0)\, h(a_0, b_0) < 1$,*
(b) *the sequences $\{a_n\}$ and $\{b_n\}$ are decreasing.*

The next aim of the study is to prove that method (1.2) is well–defined, so that we present a system of recurrence relations in the next lemma from which we obtain the last. The proof of the lemma follows from a similar way that the mentioned above and using induction.

Lemma 2.2. *If a_0 and b_0 satisfy (2.8) and $\mathbb{B}(x_0, R\, c_0\, \eta) \subseteq \mathcal{D}$, where $R = \dfrac{1}{1-\Delta}$ and $\Delta = g(a_0, b_0)\, h(a_0, b_0)$, then the next recurrence relations are true for all $n \geq 1$:*

$(\mathcal{R}\mathbf{1})$ $\| F(x_n) \| \leq g(a_{n-1}, b_{n-1}) \; \| F(x_{n-1}) \|$,
$(\mathcal{R}\mathbf{2})$ $\| B_n \| \leq h(a_{n-1}, b_{n-1}) \; \| B_{n-1} \| \leq c_n$,
$(\mathcal{R}\mathbf{3})$ $\| x_{n+1} - x_n \| \leq g(a_{n-1}, b_{n-1})\, h(a_{n-1}, b_{n-1}) \; \| B_{n-1} \| \, \| F(x_{n-1}) \|$,

$(\mathcal{R}\mathbf{4})$ $\| x_{n+1} - x_0 \| \leq \dfrac{1 - \Delta^{n+1}}{1 - \Delta} \; \| B_0 \| \, \| F(x_0) \| < R\, c_0\, \eta$,

$(\mathcal{R}\mathbf{5})$ $\| B_n \| \; \omega(\| x_{n+1} - x_n \|) \leq b_n$,
$(\mathcal{R}\mathbf{6})$ $\| I - F'(x_n)\, B_n \| \leq a_n$,
$(\mathcal{R}\mathbf{7})$ $\| I - F'(x_{n+1})\, B_n \| \leq a_n + b_n$,
$(\mathcal{R}\mathbf{8})$ $\| B_{n+1} - B_n \| \leq (a_n + b_n)\, c_n$.

Note that, from $(\mathcal{R}4)$, we obtain $x_n \in \mathcal{D}$, for all $n \geq 0$, if the hypotheses of Lemma 2.2 are satisfied.

Remark 2.3. If $a_0 = 0$, then $B_0 = (F'(x_0))^{-1}$ and the first step of iteration (1.2) is the same as in Newton's method. In this case, we have

$$\begin{aligned}
&a_1 = b_0^2, \quad b_1 = b_0\, (1 + b_0) \left(\frac{b_0\, (1 + b_0)}{1 + p} \right)^p, \quad c_1 = (1 + b_0)\, c_0,\\
&a_n = (a_{n-1} + b_{n-1})^2, \;\; n \geq 2,\\
&b_n = b_{n-1}\, g(a_{n-1}, b_{n-1})^p\, h(a_{n-1}, b_{n-1})^{1+p}, \;\; n \geq 2,\\
&c_n = c_{n-1}\, h(a_{n-1}, b_{n-1}), \;\; n \geq 2.
\end{aligned}$$

These sequences $\{a_n\}$ and $\{b_n\}$, for $n \geq 0$, are also decreasing if

$$(1 + p)\, b_0 + (1 + b_0) \left(\frac{b_0\, (1 + b_0)}{1 + p} \right)^p < 1 \quad \text{and} \quad (b_0^2 + b_1)^2 < b_0^2, \tag{2.9}$$

so that the recurrence relations appearing in Lemma 2.2 are also satisfied, excpet for ($\mathcal{R}4$), that now is

$$\| x_{n+1} - x_0 \| \leq \left(1 + f(b_0)\,\frac{1-\overline{\Delta}^n}{1-\overline{\Delta}}\right) \| B_0 \| \| F(x_0) \| < \overline{R}\, c_0\, \eta,$$

where

$$\overline{R} = 1 + f(b_0)\,\frac{b_0\,(1+b_0)}{(1+p)\,(1-\overline{\Delta})} \quad \text{and} \quad \overline{\Delta} = g(a_1, b_1)\, h(a_1, b_1).$$

Since the sequence $\{x_n\}$ is well–defined, the following aim is to see that $\{x_n\}$ is a Cauchy sequence. We then provide the following semilocal convergence result, which is also used to draw conclusions about the existence of a solution and the domain in which it is located.

Theorem 2.4. *Let $F : \mathcal{D} \subseteq \mathcal{X} \longrightarrow \mathcal{Y}$ be a Fréchet-différentiable operator on a non-empty open convex domain $\mathcal{D}$. Let $x_0 \in \mathcal{D}$ and $B_0 \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$. Suppose that conditions ($\mathcal{H}1$)-($\mathcal{H}4$), (2.8) and $\mathbb{B}(x_0, R\, c_0\, \eta) \subseteq \mathcal{D}$, are satisfied. Then the sequence $\{x_n\}$, defined by (1.2) and starting from x_0, remains in $\overline{\mathbb{B}}(x_0, R\, c_0\, \eta)$ and converges to a solution $x^\star$ of equation $F(x) = 0$.*

Remark 2.5. In the case $a_0 = 0$ ($B_0 = (F'(x_0))^{-1}$), the convergence of sequence (1.2) follows in the same way as in Theorem 2.4 with (2.9), except for R, that it now is $\overline{R}$.

In the next result we show the uniqueness of the solution $x^\star$ of equation $F(x) = 0$.

Theorem 2.6. *Suppose that conditions ($\mathcal{H}1$)-($\mathcal{H}4$) are satisfied and function ω is also strictly increasing. Then the solution $x^\star$ of equation $F(x) = 0$ is unique in the domain $\mathcal{D}_0 = \mathbb{B}(x_0, r^\star) \cap \mathcal{D}$, where $r^\star$ is the smallest positive root of the equation in the variable y:*

$$\int_{R\, c_0\, \eta}^{y} \omega_0(s)\, ds = \frac{1}{c_0}\,(1-a_0)\,(y - R\, c_0\, \eta). \tag{2.10}$$

Proof Let us assume $y^\star$ is a solution of $F(x) = 0$ in $\mathcal{D}_0$. According to Argyros ([1], [2]), we have the approximation

$$0 = F(y^\star) - F(x^\star) = \int_0^1 F'(x^\star + t\,(y^\star - x^\star))\, dt\, (y^\star - x^\star). \tag{2.11}$$

Let us set $\mathcal{M} = \displaystyle\int_0^1 F'(x^\star + t\,(y^\star - x^\star))\, dt$. We have:

$$\begin{aligned}
\| I - \mathcal{M}\, B_0 \| &\leq \| I - F'(x_0)\, B_0 \| + \| F'(x_0) - \mathcal{M} \| \| B_0 \| \\
&\leq a_0 + c_0 \int_0^1 \| F'(x_0) - F'(x^\star + t\,(y^\star - x^\star)) \| \, dt \\
&\leq a_0 + c_0 \int_0^1 \omega_0((1-t) \| x^\star - x_0 \| + t \| y^\star - x_0 \|)\, dt \\
&< a_0 + \frac{c_0}{r - R\, c_0\, \eta} \int_{R\, c_0\, \eta}^{r^\star} \omega_0(s)\, ds = 1,
\end{aligned}$$

it follows from the previous estimation and the Banach lemma of invertible operators [1], [2], that $\mathcal{M}^{-1}$ exists. In view of (2.11) we deduce $x^\star = y^\star$.

It is show in Theorem 2.4 that it is not necessary for x_0 to satisfy the conditions given by (2.8) to obtain the semilocal convergence of Ulm's method given by (1.2), since it suffices that they are satisfied for some iterate x_j of (1.2). So, we obtain the following corollary.

Corollary 2.7. *Under the conditions of Theorem 2.4, further assume: there exists $j \in \mathbb{N}$ such that*

$$g(a_j, b_j)^p\, h(a_j, b_j)^{1+p} < 1 \quad \text{and} \quad (a_j + b_j)^2 < a_j, \tag{2.12}$$

where, $a_j = \| I - F'(x_j)\, B_j \|$, $b_j = c_j\, \omega(c_j\, \overline{\eta})$, $c_j = \| B_j \|$, $\overline{\eta} = \| F(x_j) \|$ and $\mathbb{B}(x_0, R_j) \subseteq \mathcal{D}$, with $R_j = R\, c_0\, \eta + \sum_{i=0}^{i=j-1} \| x_{i+1} - x_i \|$, then sequence $\{x_n\}$, defined by (1.2) and starting from x_0, remains in $\overline{\mathbb{B}}(x_0, R_j)$ and converges to a solution $x^\star$ of equation $F(x) = 0$.

Proof The proof of Corollary 2.7 follows from the facts that the sequences $\{a_n\}$ and $\{b_n\}$ are decreasing for all $n > j$ and the recurrence relations given in Lemma 2.2 now hold for all $n > j + 1$.

In order for us to show that the R–order of convergence of method (1.2) under hypotheses ($\mathcal{H}1$)-($\mathcal{H}4$) is $1 + p$, we first need a result concerning the behavior of certain functions.

Lemma 2.8. *Let g and h be the functions given by (2.3) and (2.4) respectively and define $\delta_1 = \dfrac{a_1}{a_0}$, $\delta_2 = \dfrac{b_1}{b_0}$ and $\delta = \max\{\delta_1, \delta_2\}$. If (2.8) is satisfied, then*

(a) *$g(\delta\, x, \delta\, y) = \delta\, g(x, y)$ and $h(\delta\, x, \delta\, y) < h(x, y)$, for all $\delta \in (0, 1)$,*

(b) *$a_n < \delta^{(1+p)^{n-1}}\, a_{n-1} < \delta^{\frac{(1+p)^n - 1}{p}}\, a_0$ and $b_n < \delta^{(1+p)^{n-1}}\, b_{n-1} < \delta^{\frac{(1+p)^n - 1}{p}}\, b_0$, for all $n \geq 1$.*

We show the following result on the R–order of convergence for method (1.2):

Theorem 2.9. *Under the conditions of Theorem 2.4, the method (1.2) has R-order of convergence at least $1 + p$. Moreover, the following a priori error estimates are obtained:*

$$\| x_n - x^\star \| \leq \frac{A^n\, \delta^{\frac{(1+p)^n - 1}{p^2}}}{1 - A\, \delta^{\frac{(1+p)^n}{p}}}\, c_0\, \eta, \tag{2.13}$$

where $A = \Delta\, \delta^{-1/p}$ and $\Delta = g(a_0, b_0)\, h(a_0, b_0)$.

Remark 2.10. Observe that if F' is Lipschitz continuous in $\mathcal{D}$, then $\omega(r) = K\, r$, $K \geq 0$. and method (1.2) is of R–order of convergence at least two.

Remark 2.11. If $a_0 = 0$ ($B_0 = (F'(x_0))^{-1}$), the R–order of sequence (1.2) follows exactly as in the previous theorem.

Taking now into account the estimates regarding consecutive points are good to distance $\| x_n - x^\star \|$ (see ($\mathcal{R}$**3**) in Lemma 2.2), we can for an element x_k ($k > n$) of the sequence $\{x_n\}$ such that $\| x_k - x^\star \|$ is smaller enough and $\| x_n - x^\star \|$ can be estimated from the distance between two consecutive points. So,

$$\| x_n - x^\star \| \leq \| x_{n+j} - x^\star \| + \sum_{i=1}^{i=j} \| x_{n+i} - x_{n+i-1} \|, \quad j \geq 1, \quad n \geq 1, \tag{2.14}$$

and the error given in (2.13) is then improved.

Remark 2.12. [3] To finish, as we have indicated in the introduction, we study the convergence of the sequence–operators $\{B_n\}$. Note that $\{B_n\}$ converges to the bounded right of $F'(x^\star)$. Indeed, from ($\mathcal{R}$**8**), it follows

$$\| B_{k+1} - B_k \| \leq (a_k + b_k)\, c_k \leq (a_k + b_k)\, h(a_0, b_0)^k\, c_0,$$

since h is increasing in the both arguments $\{a_n\}$ and $\{b_n\}$ are decreasing sequences. In consequence,

$$\| B_{k+1} - B_k \| \leq \delta^{\frac{(1+p)^k - 1}{p}} (a_0 + b_0)\, h(a_0, b_0)^k\, c_0.$$

Therefore,

$$\begin{aligned} \| B_{n+m} - B_n \| &\leq \left(\sum_{k=0}^{k=m-1} \delta^{\frac{(1+p)^{n+k} - 1}{p}} h(a_0, b_0)^{n+k} \right) (a_0 + b_0)\, c_0 \\ &\leq \delta^{-1/p} (a_0 + b_0)\, c_0\, h(a_0, b_0)^{n+m-1}\, S \end{aligned}$$

where

$$S = \sum_{k=0}^{k=m-1} \delta^{\frac{(1+p)^{n+k}}{p}}.$$

Moreover,

$$S \leq \delta^{\frac{(1+p)^{n+m-1}}{p}} \left(\delta^{\frac{(1+p)^n}{p}(1-(1+p)^{m-1})} \frac{1 - \delta^{m\,(1+p)^n}}{1 - \delta^{(1+p)^n}} \right),$$

since $\delta^{\frac{(1+p)^k}{p}} \leq \delta^{\frac{(1+p)^n}{p}}\, \delta^{(1+p)^n\,(k-n)}$, for $k = n+1, n+2, \cdots, n+m-1$. Thus, $\{B_n\}$ is a Cauchy sequence and then $\lim_n B_n = B^\star$. On the other hand, $\| I - F'(x^\star)\, B_n \| \longrightarrow 0$ by letting $n \to \infty$ and taking into account that

$$\| I - F'(x_n)\, B_n \| \leq a_n \leq \delta^{2\,((1+p)^n - 1)/p}\, a_1,$$

$$\| B_n \| \leq h(a_0, b_0)^n\, c_0,$$

$$\| F'(x^\star) - F'(x_n) \| \leq \left(\frac{\Delta^n}{1 - \Delta} \right)^p \omega(\eta).$$

Consequently, $B^\star$ is the bounded right inverse of $F'(x^\star)$.

Remark 2.13. The sufficient convergence conditions given in [3] corresponding to (2.8) and (2.9) are given by

$$g(\alpha_0, \beta_0)^p\, h(\alpha_0, \beta_0)^{1+p} < 1 \quad \text{and} \quad (\alpha_0 + \beta_0)^2 < \alpha_0, \tag{2.15}$$

and

$$(1+p)\, \beta_0 + (1 + \beta_0) \left(\frac{\beta_0\, (1 + \beta_0)}{1 + p} \right)^p < 1 \quad \text{and} \quad (\beta_0^2 + \beta_1^2)^2 < \beta_0^2, \tag{2.16}$$

respectively.

In case strict inequality holds in (2.1), conditions (2.15) and (2.16) imply (2.8) and (2.9) respectively but not necessary vice verca (unless if $\omega_0(r) = \omega(r)$ for all $r > 0$). Moreover due to the fact that

$$b_0 \leq \beta_0;$$

the rest of the advantages already stated at the introduction of this study hold true.

We provide a numerical example to show that our conditions (2.8) (or (2.9)) hold, whereas (2.15) (or (2.16)) do not.

Example 2.14. $\mathcal{X} = \mathcal{Y} = \mathbb{R}$, $\mathcal{D} = [q, 2-q]$, $q \in [0,1]$, $x_0 = 1$ and define function F on $\mathcal{D}$ by

$$F(x) = x^3 - q. \tag{2.17}$$

Using (2.17), ($\mathcal{H}1$)-($\mathcal{H}4$), (2.2), (2.4) and (2.8), we get

$$\eta = 1-q, \quad \omega(r) = 6\,(2-q)\,r, \quad \omega_0(r) = 3\,(3-q)\,r, \quad b_0 = 3\,c_0^2\,(3-q)\,(1-q)$$

and

$$\beta_0 = 6\,c_0^2(2-q)\,(1-q).$$

Let $c_0 = B_0 = 8/30$ and $q = 0.55$.
Then we obtain

$$\alpha_0 = a_0 = 0.2, \quad \beta_0 = 0.2784, \quad b_0 = 0.2448 \quad \text{and} \quad (\alpha_0+\beta_0)^2 = 0.22886656 > 0.2.$$

That is there is no guarantee that method (1.2) converges to $x^\star = \sqrt[3]{q} = 0.819321271$, since (2.15) is violated.

Howeover conditions (2.8) and (2.9) are satisfied since they become

$$0.672992926 < 1 \quad \text{and} \quad 0.19784704 < 0.2,$$

respectively.

Hence, the conclusions of Theorem 2.4 for equation apply and our method (1.2) converges to $x^\star$.

Remark 2.15. The earlier results on method (1.2), [4], [6], [7]-[10] require that operator F' satisfies the Lipschitz condition:

$$\| F'(x) - F'(y) \| \leq K \ \| x - y \| \quad \text{for all } x, y \in \mathcal{D}. \tag{2.18}$$

It follows from (2.18) that there exists K_0 such that

$$\| F'(x) - F'(x_0) \| \leq K_0 \ \| x - x_0 \| \quad \text{for all } x \in \mathcal{D}. \tag{2.19}$$

Clearly

$$K_0 \leq K \tag{2.20}$$

holds and $\dfrac{K}{K_0}$ can be arbitrarily large [1], [2].

In case strict inequality holds in (2.20), one can visit the results mentioned above and use (2.18) and (2.19) instead of only (2.18) in the convergence analysis of method (1.2). It then follows that the resulting approach will produce a finer convergence analysis for method (1.2) with advantages over earlier works as stated in the introduction of this study. Howeover we leave the details to the motivated reader.

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