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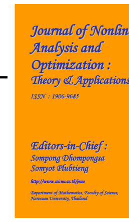

POSITIVE SOLUTIONS AND EIGENVALUES FOR SECOND ORDER INTEGRAL BOUNDARY VALUE PROBLEMS WITH SIGN-CHANGING NONLINEARITIES

YUMEI ZOU*

Department of Statistics and Finance, Shandong University of Science and Technology, 266590, People's Republic of China

ABSTRACT. We investigate eigenvalue intervals for the existence of positive solutions for the second order integral boundary value problem

$$\begin{cases} u''(t) + \lambda f(u(t)) = 0, \quad t \in (0,1), \\ u(0) = \displaystyle\int_0^1 u(s)d\alpha(s), \quad u(1) = \displaystyle\int_0^1 u(s)d\beta(s) \end{cases}$$

where $f \in C(\mathbb{R}, \mathbb{R})$ is sign-changing.

KEYWORDS : Positive solution; Sign-changing nonlinearities; Existence.
AMS Subject Classification: 34B10 34B18.

1. INTRODUCTION

In this paper, we are concerned with determining values of λ (eigenvalues), for which exist positive solutions of nonlinear second-order integral boundary-value problem

$$\begin{cases} u''(t) + \lambda f(u(t)) = 0, \quad t \in (0,1), \\ u(0) = \displaystyle\int_0^1 u(s)d\alpha(s), \quad u(1) = \displaystyle\int_0^1 u(s)d\beta(s), \end{cases} \tag{1.1}$$

where $f \in C(\mathbb{R}, \mathbb{R})$; α and β are right continuous on $[0, 1)$, left continuous at $t = 1$, and nondecreasing on [0,1], with $\alpha(0) = \beta(0) = 0$; $\int_0^1 u(s)d\alpha(s)$ and $\int_0^1 u(s)d\beta(s)$ denote the Riemann-Stieltjes integrals of u with respect to α and β, respectively. We shall concentrate on the case when the nonlinearity $f(u)$ is allowed to change sign, which is of particular mathematical interest.

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* *Corresponding author.*
Email address : sdzouym@126.com(YUMEI ZOU).
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The motivation for the present work stems from many recent investigations. In fact, boundary value problems with integral boundary conditions for ordinary differential equations arises in different areas of applied mathematics and physics. For example, heat conduction, chemical engineering, underground water flow, thermo-elasticity, and plasma physics can be reduced to the nonlocal problems with integral boundary conditions. Concerning the existence of positive solutions for eigenvalue problem, we refer the reader to [4, 5, 7, 8]. For the special case $\lambda = 1$, (1.1) have been the subject matter of many recent publications on singular boundary value problems, for this we refer the reader to the papers by Karakostas and Tsamatos [10, 11], Yuhua Li and Fuyi Li [12], Webb and Infante [16, 17], Yang [18, 19, 20] and Zhang and Sun [22] and the references therein. For more information about the general theory of integral equations and their relation with boundary value problems we refer to the book of Corduneanu [6] and Agarwal and O'Regan [3]

Positive solutions are usually the ones of interest and because of the difficulties associated with proving the existence of such solutions using the techniques of nonlinear functional analysis, most of the recent work assumes nonnegativity of $f(u)$ in order to generate positive operators using the positivity of Green's function. To the authors's knowledge, there are few papers that have considered the existence of positive solutions for local, nonlocal and, particularly, integral boundary value problems involving sign-changing nonlinearities (see[1, 2, 13, 14, 15]).

Recently, using the a priori estimate method and the Leray-Schauder fixed point theorem, Yang [18] studied (1.1) with $\lambda = 1$ under the assumptions:

(H_1) $f(x) > 0, x \in (-\infty, 0]$ and there is $p > 0$ such that $f(x) < 0, x \in (p, +\infty)$.

(H_2) $\kappa_1 > 0, \kappa_4 > 0, \kappa = \kappa_1\kappa_4 - \kappa_2\kappa_3 > 0$, where

$$\kappa_1 = 1 - \int_0^1 (1-t)d\alpha(t), \quad \kappa_2 = \int_0^1 td\alpha(t),$$

$$\kappa_3 = \int_0^1 (1-t)d\beta(t), \quad \kappa_4 = 1 - \int_0^1 td\beta(t).$$

Motivated by [18], the purpose of this paper is to give sufficient conditions on $f(u)$ to determine ranges of λ for which positive solution exists. We make the following assumptions

(H_1') $f : \mathbb{R} \to \mathbb{R}$ is continuous and there is $p > 0$ such that $f(x) > 0,\ x \in (0, p)$ and $f(p) = 0$.

(H_2') $\int_0^1 d\alpha(t) < 1, \ \int_0^1 d\beta(t) < 1$.

Remark 1.1. It is obvious that the condition imposed on f in this paper is weaker than that of [18]. But the condition about $\alpha(t), \beta(t)$ is stronger than that of [18].

2. PRELIMINARIES

Let X be the Banach space $C[0,1]$ with $\|u\| = \sup\limits_{t\in[0,1]} |u(t)|$. Define a set $K \subset X$ by

$$K = \{u \in X : u(t) \geq t(1-t)\|u\|, \ \ t \in [0,1]\}.$$

It can be easily verified that K is indeed a cone in X. For any $r > 0$, defined Ω_r by $\Omega_r = \{u \in K : \ \|u\| < r\}$.

To study (1.1), consider the map $T : X \to X$ defined by

$$(Tu)(t) = \int_0^1 k(t,s)u(s)ds + \kappa^{-1}(1-t,t)\begin{pmatrix} \kappa_4 & \kappa_2 \\ \kappa_3 & \kappa_1 \end{pmatrix}\begin{pmatrix} \int_0^1 d\alpha(t)\int_0^1 k(t,s)u(s)ds \\ \int_0^1 d\beta(t)\int_0^1 k(t,s)u(s)ds \end{pmatrix}$$

and $F : X \to X$ defined by

$$(F(u))(t) = f(u(t)),$$

where $k(t,s)$ is given by

$$k(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1. \end{cases}$$

Then if (H_1) and (H_2) (or (H_1') and (H_2')) hold, by [18, 21], u is a solution of (1.1) if and only if $u \in C^2[0,1]$ is a solution of the equation

$$(I - \lambda TF)u = 0, \text{ that is, a fixed point of } \lambda TF,$$

where by standard arguments, $TF : X \to X$ is a compact map.

For the function $k(t,s)$, it is easy to know that

$$t(1-t)s(1-s) \le k(t,s) \le s(1-s), \quad t,s \in [0,1]. \tag{2.1}$$

Lemma 2.1. *$T(K) \subset K$ and the map $T : K \to K$ is completely continuous.*

Proof. The inequality (2.1) and the definition of T imply that $T(K) \subset K$. The complete continuity of the integral operator T is well known. □

The following lemma is needed in our argument.

Lemma 2.2. [9] *Let X be a Banach space and K a cone in X. Assume that $T : \Omega_r \to K$ is completely continuous such that $Tu \neq u$ for $u \in \partial\Omega_r$.*
(i) If $\|Tu\| \ge \|u\|$ for $u \in \partial\Omega_r$, then $i(T, \Omega_r, K) = 0$.
(ii) If $\|Tu\| \le \|u\|$ for$u \in \partial\Omega_r$, then $i(T, \Omega_r, K) = 1$.

3. MAIN RESULTS

Our first result is the following theorem in which we shall prove existence of at least a solution of the BVP(1.1).

Theorem 3.1. *If (H_1') and (H_2') hold, then for $\lambda > 0$, (1.1) has at least a solution $u \in C^2[0,1]$.*

Proof. We define a auxiliary functions $\widetilde{f} : \mathbb{R} \to \mathbb{R}$ by

$$\widetilde{f}(u) = \begin{cases} f(0), & u \le 0, \\ f(u), & 0 < u < p, \\ 0, & u \ge p. \end{cases}$$

Thus $\widetilde{f}$ is continuous and bounded on $\mathbb{R}$ so there exists $M > 0$ such that

$$|\widetilde{f}| \le M, \text{ on } \mathbb{R}.$$

Consider the modified problem

$$\begin{cases} u''(t) + \lambda\widetilde{f}(u(t)) = 0, & t \in (0,1), \\ u(0) = \int_0^1 u(s)d\alpha(s), & u(1) = \int_0^1 u(s)d\beta(s). \end{cases} \tag{3.1}$$

This is equivalent to the integral equation

$$\begin{aligned} u(t) &= \lambda \int_0^1 k(t,s)\widetilde{f}(u(s))ds \\ &+\frac{\lambda}{\kappa}(1-t,t)\begin{pmatrix} \kappa_4 & \kappa_2 \\ \kappa_3 & \kappa_1 \end{pmatrix}\begin{pmatrix} \int_0^1 d\alpha(t)\int_0^1 k(t,s)\widetilde{f}(u(s))ds \\ \int_0^1 d\beta(t)\int_0^1 k(t,s)\widetilde{f}(u(s))ds. \end{pmatrix} \end{aligned} \tag{3.2}$$

We write (3.2) as an operator equation

$$(I - T_\lambda)u = 0,$$

where $T_\lambda = \lambda T\widetilde{F}$, and $\widetilde{F} : X \to X$ defined by

$$(\widetilde{F}(u))(t) = \widetilde{f}(u(t))$$

is bounded. Notice that $|\widetilde{f}| \leq M$. Thus for some constant $N > 0$, independent of λ, u, we have

$$\|T_\lambda u\|_{C[0,1]} \leq \lambda MN.$$

It follows from the Schauder fixed point theorem that T_λ has a fixed point $u_\lambda \in C[0,1]$. This together with the equation in (3.1) implies that $u_\lambda \in C^2[0,1]$.

Since results to be proved in this Theorem are true for any positive parameter λ. So in the rest of proof, we write u_λ as u for simplicity.

To finish the proof from the definition of $\widetilde{f}$, it suffices to show that any solution u of (3.1) satisfies

$$0 \leq u(t) \leq p, \quad t \in (0,1),$$

i.e., any solution u of (3.1) in fact is a solution of (1.1). Now we claim that any solution u of (3.1) satisfies $0 \leq u(t) \leq p,\ t \in (0,1)$. Firstly, we show that $u \geq 0,\ t \in [0,1]$. In fact, the nonnegativity of $\widetilde{f}$ ensures that

$$u''(t) \leq 0,\ t \in [0,1]. \tag{3.3}$$

So, in order to get the desired results, we only need to prove that $u(0) \geq 0$ and $u(1) \geq 0$. (3.3) implies

$$u(t) \geq (1-t)u(0) + tu(1), \quad t \in [0,1].$$

Therefore,

$$u(0) = \int_0^1 u(t)d\alpha(t) \geq u(0)\int_0^1 (1-t)d\alpha(t) + u(1)\int_0^1 td\alpha(t)$$

and

$$u(1) = \int_0^1 u(t)d\beta(t) \geq u(0)\int_0^1 (1-t)d\beta(t) + u(1)\int_0^1 td\beta(t).$$

The last two inequalities can be written as

$$\begin{pmatrix} \kappa_1 & -\kappa_2 \\ -\kappa_3 & \kappa_4 \end{pmatrix}\begin{pmatrix} u(0) \\ u(1) \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Then by Remark 1.1, we have

$$\kappa_1 > 0,\ \kappa_2 \geq 0,\ \kappa_3 \geq 0,\ \kappa_4 > 0,\ \kappa > 0.$$

Thus

$$\begin{pmatrix} u(0) \\ u(1) \end{pmatrix} \geq \frac{1}{\kappa}\begin{pmatrix} \kappa_4 & \kappa_3 \\ \kappa_2 & \kappa_1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Next, we show that $u(t) \le p,\ t \in [0,1]$. If not, then there exists a $t_0 \in [0,1]$ such that $u(t_0) = \max_{t\in[0,1]} u(t) > p$.

If $t_0 = 0$, then $u(0) = \int_0^1 u(t)d\alpha(t) \le u(0)\int_0^1 d\alpha(t)$, from (H_2'), we have the contradiction $u(0) \le 0$. Similarly, the condition (H_2') gives $t_0 \ne 1$.

If $t_0 \in (0,1)$, then we have $u'(t_0) = 0$. We consider the following two cases: case (i), $u(t) \ge p,\ \ t \in [0,1]$; case (ii), there exists $t_1 \in [0,1]$ such that $u(t_1) < p$. In the second case , without loss of generality, we assume that $t_1 \in [0,t_0)$ and $u(t) > p,\ \ t \in (t_1,t_0]$.

For case (i), we have $u''(t) = -\lambda\widetilde{f}(u(t)) = -\lambda f(p) = 0,\ t \in (0,1)$. Thus, u' is constant on $[0,1]$. Since $u'(t_0) = 0$, it follows that $u'(t) = 0$ for $t \in [0,1]$. Consequently, $u(t) \equiv u(t_0) > p$ on $[0,1]$. On the other hand, we have $u(0) = \int_0^1 u(t)d\alpha(t) \le u(0)\int_0^1 d\alpha(t)$, from (H_2'), we obtain

$$u(t) \equiv u(0) \le 0.$$

Which is a contradiction.

For case (ii), we have $u''(t) = -\lambda\widetilde{f}(u(t)) = -\lambda f(p) = 0,\ t \in (t_1,t_0]$. This together with $u'(t_0) = 0$ implies that $u(t_1) = u(t_0) > p$, contradicting $u(t_1) < p$. □

When $f(0) > 0$, the existence results in Theorem 3.1 can be improved in

Theorem 3.2. *If (H_1') and (H_2') hold with $f(0) > 0$, then for $\lambda > 0$, (1.1) has at least one positive solution $u \in C^2[0,1]$.*

Remark 3.3. If $f(0) = 0$, then Theorem 3.2 may be false, as the following counterexample shows.

$$\begin{cases} -u''(t) = \lambda u(t)(1-u(t)), & t \in (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{3.4}$$

Suppose u is a positive solution of (3.4). Multiplying the equation in (3.4) by $\sin \pi t$ and integrating on $[0,1]$, we have

$$\begin{aligned} \pi^2 \int_0^1 u(t)\sin \pi t dt &= -\int_0^1 u''(t)\sin \pi t dt \\ &= \lambda \int_0^1 u(t)(1-u(t))\sin \pi t dt < \lambda \int_0^1 u(t)\sin \pi t dt. \end{aligned}$$

Thus, for every $\lambda \le \pi^2$, Theorem 3.2 is not true. However, Theorem 3.2 remain true for sufficiently large λ.

Theorem 3.4. *If (H_1') and (H_2') hold with $f(0) = 0$. Then exists a $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$, (1.1) has at least one positive solution $u \in C^2[0,1]$.*

Proof. Using Theorem 3.1, we can show that any solution of the modified problem (3.1) satisfies

$$0 \le u(t) \le p, \quad t \in [0,1]$$

and hence is a solution of (1.1). So it suffices to show that the compact operator T_λ has a nonzero fixed point in $K\backslash\{\theta\}$.

Take $r \in (0,p)$. Let $\lambda_0 = \dfrac{512r}{11m}$, where $m = \min_{v\in[\frac{r}{4},r]} f(v) > 0$. Notice that for any $u \in K$ we have that $u(t) \ge t(1-t)\|u\|$ for all $t \in [0,1]$. In particular, we have

$u(t) \geq \frac{3}{16}\|u\|$ for all $t \in [\frac{1}{4}, \frac{3}{4}]$. Let $u \in \partial\Omega_r$, then $f(u(t)) \geq m$ for $t \in [\frac{1}{4}, \frac{3}{4}]$. Hence for $\lambda > \lambda_0$, we have

$$\begin{aligned}\|T_\lambda u\| &\geq \lambda \min_{t\in[\frac{1}{4},\frac{3}{4}]} \int_{\frac{1}{4}}^{\frac{3}{4}} k(t,s)f(u(s))ds \geq \lambda \min_{t\in[\frac{1}{4},\frac{3}{4}]} \left(t(1-t)\int_{\frac{1}{4}}^{\frac{3}{4}} s(1-s)f(u(s))ds \right) \\ &\geq \frac{3\lambda}{16}\int_{\frac{1}{4}}^{\frac{3}{4}} s(1-s)f(u(s))ds \\ &\geq \frac{11}{512}\lambda m > r = \|u\|, \quad u \in \partial\Omega_r.\end{aligned}$$

On the other hand, for each fixed $\lambda > \lambda_0$ since $\widetilde{f}(u)$ is bounded, there is an $R > r$ such that

$$\|T_\lambda u\| < R = \|u\|, \quad u \in \partial\Omega_R.$$

It follows from Lemma 2.2 that

$$i(T_\lambda, \Omega_r, K) = 0, \text{ while } i(T_\lambda, \Omega_R, K) = 1,$$

and hence,

$$i(T_\lambda, \Omega_R \backslash \overline{\Omega_r}, K) = 1.$$

Thus, T_λ has a fixed point u in $\Omega_R\backslash\overline{\Omega_r}$. Theorem 3.1 implies that the fixed point u is a solution of (1.1) such that

$$0 < r < \|u\| \leq p.$$

Consequently, (1.1) has at least a positive solution u for each $\lambda > \lambda_0$. □

References

1. R. P. Agarwal, H. Gao, D. O'Regan, X. Zhang, Existence principles and an upper and lower solution theory for the one-dimensional p-Laplacian singular boundary value problem with sign changing nonlinearities, Dynam. Systems Appl. 15(2006), 3-19.
2. R. P. Agarwal, D. O'Regan, Existence criteria for singular boundary value problem with sign changing nonlinearities, J. Differential Equations. 183(2002), 409-433.
3. R. P. Agarwal, D. O'Regan, Infinite Interval Problems for Differential, Difference and Integral Equations, Kluwer Academic Publishers, Dordrecht, 2001.
4. R. P. Agarwal, M. Bohner, P. J. Y. Wong, Positive solutions and eigenvalues of conjugate boundary value problems, Proceedings of the Edinburgh Mathematical Society, 42(1999)02, 349-374.
5. A. Castro, C. Maya, R. Shivaji, Nonlinear eigenvalue problems with semipositone structure, Nonlinear Differential Equations, Electron. J. Diff. Eqns., Conf. 05, 2000, pp. 33-49
6. C. Corduneanu, Integral Equations and Applications, Cambridge University Press, Cambridge, 1991.
7. P. W. Eloe, J. Henderson, Positive solutions and nonlinear multipoint conjugate eigenvalue problems, Electronic Journal of Differential Equations, 03(1997), 1-11.
8. A. M. Fink, J. A. Gatica, G. E. Hernansez, Eigenvalues of generalized Gel'fand models, Nonlinear Anal. 20 (1993), 1453-1468.
9. D. Guo and V. Lakshmikantham, Nonlinear Problems in Abstract Cones, Academic Press, New York, 1988.
10. G. L. Karakostas, P.Ch. Tsamatos, Multiple positive solutions of some Fredholm integral equations arisen from nonlocal boundary-value problems, Electron. J. Differential Equations. 30(2002), 1-17.
11. G. L. Karakostas, P.Ch. Tsamatos, Existence of multiple positive solutions for a nonlocal boundary value problem, Topol. Methods Nonlinear Anal. 19(2002), 109-121.
12. Y. Li, F. Li, Sign-changing solutions for second order integral boundary value problems, Nonlinear Anal. 69(2008), 1179-1187.
13. Y. Li, Positive solutions for second order boundary value problems with sign changing nonlinear term, J. Math. Anal. Appl. 282(2003), 232-240.
14. B. Liu, Positive solutions of second-order three-point boundary value problems with change of sign, Comput. Math. Appl. 47(2004), 1351-1361.
15. J. Ren, W. Ge, Positive solution for three-point boundary value problems with sign changing nonlinearities, Appl. Math. Lett. 17(2004), 451-458.

16. J. R. L. Webb, G. Infante, Positive solutions of nonlocal boundary value problems involving integral conditions. NoDEA Nonlinear Differential Equations Appl. 15(2008), 45-67.
17. J. R. L. Webb, G. Infante, Positive solutions of nonlocal boundary value problems: a unified approach. J. London Math. Soc. 74(2006), 673-693.
18. Z. Yang, Existence and uniqueness of positive solutions for an integral boundary value problem, Nonlinear Analysis. 69(2008):3910-3918.
19. Z. Yang, Existence and nonexistence results for positive solutions of an integral boundary value problem, Nonlinear Anal. 65(2006), 1489-1511.
20. Z. Yang, Positive solutions of a second-order integral boundary value problem, J. Math. Anal. Appl. 321(2006), 751-765.
21. Z. Yang, Existence of nontrivial solutions for a nonlinear Sturm-Liouville problem with integral boundary conditions, Nonlinear Analysis, 68(2008), 216-225.
22. X. Zhang, J. Sun, On multiple sign-changing solutions for some second-order integral boundary value problems, Electronic Journal of Qualitative Theory of Differential Equations 44(2010), 1-15.