
FIXED POINT AND MEAN CONVERGENCE THEOREMS FOR A FAMILY OF λ -HYBRID MAPPINGS

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ABSTRACT. The aim of this paper is to prove fixed point and mean convergence theorems for a sequence of λ -hybrid mappings in Hilbert spaces.

KEYWORDS : λ -hybrid mapping; Fixed point; Mean convergence theorem.

1. INTRODUCTION

In this paper, we show fixed point and mean convergence theorems for a sequence of λ -hybrid mappings in Hilbert spaces. Particularly, we focus on pointwise convergent sequences of such mappings.

According to [2] and §2, every nonexpansive mapping [5, 6, 10] is a 1-hybrid mapping and every nonspreading mapping introduced by Kohsaka and Takahashi [7] is a 0-hybrid mapping. Thus our results may be regarded as generalizations of results of [1] and [8]. Akatsuka, Aoyama, and Takahashi [1] showed a mean convergence theorem for a pointwise convergent sequence of nonexpansive mappings; Kurokawa and Takahashi [8] proved some mean convergence theorems for nonspreading mappings in Hilbert spaces.

Moreover, since the convex combination of the identity mapping and a strictly pseudononspreading mapping introduced by Osilike and Isiogugu [9] is λ -hybrid for some real number λ , our mean convergence theorem is a generalization of [9]. Osilike and Isiogugu [9] showed some mean convergence theorems for strictly pseudononspreading mappings in Hilbert spaces.

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2. PRELIMINARIES

Throughout the present paper, H denotes a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$, C a nonempty closed convex subset of H , I the identity mapping on H , and \mathbb{N} the set of positive integers. Strong convergence of a sequence $\{x_n\}$ in H to x is denoted by $x_n \rightarrow x$ and weak convergence by $x_n \rightharpoonup x$. The metric projection of H onto C is denoted by P_C , that is, for each $x \in H$, $P_C x$ is the unique point in C such that $\|P_C x - x\| = \min\{\|y - x\| : y \in C\}$. It is known that P_C is nonexpansive and

$$\langle y - P_C x, x - P_C x \rangle \leq 0 \quad (2.1)$$

for all $x \in H$ and $y \in C$; see [10].

Let D be a nonempty subset of H . The set of fixed points of a mapping $T: D \rightarrow H$ is denoted by $F(T)$. A mapping $T: D \rightarrow H$ is said to be quasi-nonexpansive if $F(T)$ is nonempty and $\|Tx - z\| \leq \|x - z\|$ for all $x \in D$ and $z \in F(T)$. Let λ be a real number. A mapping $T: D \rightarrow H$ is said to be λ -hybrid [2] if

$$2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2 - 2\lambda\langle x - Tx, y - Ty \rangle$$

or equivalently

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2(1 - \lambda)\langle x - Tx, y - Ty \rangle$$

for all $x, y \in D$. Let κ be a real number with $\kappa \in [0, 1)$. A mapping $T: D \rightarrow H$ is said to be κ -strictly pseudononspreading [9] if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty \rangle + \kappa\|x - Tx - (y - Ty)\|^2$$

for all $x, y \in D$. It is known that

- T is λ -hybrid for every $\lambda \in [0, 1]$ if T is a firmly nonexpansive mapping [3, 4, 5, 6];
- T is 1-hybrid if and only if T is nonexpansive;
- T is 0-hybrid if and only if T is nonspreading in the sense of [7];
- T is 1/2-hybrid if and only if T is hybrid in the sense of [11];
- $F(T)$ is closed and convex if $T: C \rightarrow H$ is a quasi-nonexpansive mapping;
- T is quasi-nonexpansive if T is a λ -hybrid mapping with a fixed point.

The following lemma plays an important role in the present paper.

Lemma 2.1. *Let H be a Hilbert space, D a nonempty subset of H , γ and κ real numbers, and $T: D \rightarrow H$ a mapping such that*

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\gamma\langle x - Tx, y - Ty \rangle + \kappa\|x - Tx - (y - Ty)\|^2 \quad (2.2)$$

for all $x, y \in D$. Let $T_\alpha: D \rightarrow H$ be a mapping defined by $T_\alpha = \alpha I + (1 - \alpha)T$, where α is a real number with $\alpha < 1$. Then

$$\begin{aligned} \|T_\alpha x - T_\alpha y\|^2 &\leq \frac{\alpha - \kappa}{1 - \alpha} \|x - T_\alpha x - (y - T_\alpha y)\|^2 \\ &\leq \|x - y\|^2 + \frac{2\gamma}{1 - \alpha} \langle x - T_\alpha x, y - T_\alpha y \rangle \end{aligned} \quad (2.3)$$

for all $x, y \in D$. Moreover, if $\kappa \leq \alpha$, then T_α is $(1 - \alpha - \gamma)/(1 - \alpha)$ -hybrid.

Proof. Let $x, y \in D$ be fixed. Since $1 - \alpha > 0$ and $I - T = (I - T_\alpha)/(1 - \alpha)$, it follows from (2.2) that

$$\begin{aligned} (1 - \alpha)\|Tx - Ty\|^2 &\leq (1 - \alpha)(\|x - y\|^2 + 2\gamma\langle x - Tx, y - Ty \rangle + \kappa\|x - Tx - (y - Ty)\|^2) \end{aligned}$$

$$\begin{aligned}
&= (1 - \alpha)\|x - y\|^2 \\
&\quad + \frac{2\gamma}{1 - \alpha}\langle x - T_\alpha x, y - T_\alpha y \rangle + \frac{\kappa}{1 - \alpha}\|x - T_\alpha x - (y - T_\alpha y)\|^2
\end{aligned}$$

and hence

$$\begin{aligned}
&\|T_\alpha x - T_\alpha y\|^2 \\
&= \alpha\|x - y\|^2 + (1 - \alpha)\|Tx - Ty\|^2 - \alpha(1 - \alpha)\|x - Tx - (y - Ty)\|^2 \\
&= \alpha\|x - y\|^2 + (1 - \alpha)\|Tx - Ty\|^2 - \frac{\alpha}{1 - \alpha}\|x - T_\alpha x - (y - T_\alpha y)\|^2 \\
&\leq \|x - y\|^2 + \frac{2\gamma}{1 - \alpha}\langle x - T_\alpha x, y - T_\alpha y \rangle - \frac{\alpha - \kappa}{1 - \alpha}\|x - T_\alpha x - (y - T_\alpha y)\|^2.
\end{aligned}$$

Thus (2.3) holds. Now we suppose that $\kappa \leq \alpha$. Then $(\alpha - \kappa)/(1 - \alpha) \geq 0$, and (2.3) yields that

$$\|T_\alpha x - T_\alpha y\|^2 \leq \|x - y\|^2 + 2 \left(1 - \frac{1 - \alpha - \gamma}{1 - \alpha}\right) \langle x - T_\alpha x, y - T_\alpha y \rangle.$$

Therefore, T_α is $(1 - \alpha - \gamma)/(1 - \alpha)$ -hybrid. \square

Lemma 2.1 implies the following lemma.

Lemma 2.2. *Let H be a Hilbert space, D a nonempty subset of H , λ a real number, and $T: D \rightarrow H$ a λ -hybrid mapping. Let $T_\alpha: D \rightarrow H$ be a mapping defined by $T_\alpha = \alpha I + (1 - \alpha)T$, where α a real number with $0 \leq \alpha < 1$. Then T_α is $(\lambda - \alpha)/(1 - \alpha)$ -hybrid.*

Proof. Assuming that $\gamma = 1 - \lambda$ and $\kappa = 0$ in Lemma 2.1, we obtain the conclusion. \square

Using Lemma 2.2, we can show the following corollary.

Corollary 2.3. *Let H be a Hilbert space and D a nonempty convex subset of H . Suppose that every nonspreading self-mapping on D has a fixed point. If $\lambda \in [0, 1)$, then every λ -hybrid mapping $T: D \rightarrow D$ has a fixed point.*

Proof. Let $\lambda \in [0, 1)$ and let $T: D \rightarrow D$ be a λ -hybrid mapping. Then it follows from Lemma 2.2 that $T_\lambda = \lambda I + (1 - \lambda)T$ is a nonspreading mapping of D into itself. Hence, by assumption, we know that $F(T_\lambda)$ is nonempty. On the other hand, it obviously holds that $F(T_\lambda) = F(T)$. Thus $F(T)$ is nonempty. \square

Remark 2.4. It is known that every nonspreading self-mapping on C has a fixed point if C is a nonempty bounded closed convex subset of H ; see [7, Theorem 4.1].

Lemma 2.1 also implies the following lemma, which was essentially proven in [9].

Lemma 2.5. *Let H be a Hilbert space, D a nonempty subset of H , κ and β real numbers with $0 \leq \kappa \leq \beta < 1$, $T: D \rightarrow H$ a κ -strictly pseudononspreading mapping, and $T_\beta: D \rightarrow H$ the mapping defined by $T_\beta = \beta I + (1 - \beta)T$. Then T_β is $-\beta/(1 - \beta)$ -hybrid.*

Proof. Assuming that $\alpha = \beta$ and $\gamma = 1$ in Lemma 2.1, we obtain the conclusion. \square

We need the following lemmas in order to prove our results in the remainder sections.

Lemma 2.6. Let $\{x_n\}$ and $\{y_n\}$ be sequences in a Hilbert space H and $\{\eta_n\}$ a sequence of real numbers. Suppose that $\{x_n\}$ is bounded and both $\{y_n\}$ and $\{\eta_n\}$ are convergent. Then

$$\frac{1}{n} \sum_{k=1}^n \eta_k \langle x_{k+1} - x_k, y_k \rangle \longrightarrow 0$$

as $n \longrightarrow \infty$.

Proof. Let y and η be the limits of $\{y_n\}$ and $\{\eta_n\}$, respectively. Since $\{x_n\}$ is bounded, it follows that $\langle x_{n+1} - x_n, y_n - y \rangle \longrightarrow 0$ and hence

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^n \langle x_{k+1} - x_k, y_k \rangle &= \frac{1}{n} \sum_{k=1}^n \langle x_{k+1} - x_k, y \rangle + \frac{1}{n} \sum_{k=1}^n \langle x_{k+1} - x_k, y_k - y \rangle \\ &= \frac{1}{n} \langle x_{n+1} - x_1, y \rangle + \frac{1}{n} \sum_{k=1}^n \langle x_{k+1} - x_k, y_k - y \rangle \longrightarrow 0 \end{aligned}$$

as $n \longrightarrow \infty$. Therefore, since $\{\langle x_{n+1} - x_n, y_n \rangle\}$ is bounded, it follows that

$$\frac{1}{n} \sum_{k=1}^n \eta_k \langle x_{k+1} - x_k, y_k \rangle = \frac{1}{n} \sum_{k=1}^n \eta \langle x_{k+1} - x_k, y_k \rangle + \frac{1}{n} \sum_{k=1}^n (\eta_k - \eta) \langle x_{k+1} - x_k, y_k \rangle \longrightarrow 0$$

as $n \longrightarrow \infty$. \square

The following lemma was essentially shown in [1, Lemma 3.1], where $\{\xi_n\}$ was assumed to be convergent to 0. For the sake of completeness, we give the proof.

Lemma 2.7. Let H be a Hilbert space, C a nonempty closed convex subset of H , and $T: C \longrightarrow H$ a mapping. Let $\{x_n\}$ be a sequence in C , $\{\xi_n\}$ a sequence of real numbers, $\{z_n\}$ a sequence in C defined by $z_n = (1/n) \sum_{k=1}^n x_k$ for $n \in \mathbb{N}$, and z a weak cluster point of $\{z_n\}$. Suppose that

$$\xi_n \leq \|x_n - z\|^2 - \|x_{n+1} - Tz\|^2$$

for every $n \in \mathbb{N}$ and $(1/n) \sum_{k=1}^n \xi_k \longrightarrow 0$ as $n \longrightarrow \infty$. Then z is a fixed point of T .

Proof. By assumption, it is clear that

$$\begin{aligned} \xi_k &\leq \|x_k - z\|^2 - \|x_{k+1} - Tz\|^2 \\ &= \|x_k - Tz + Tz - z\|^2 - \|x_{k+1} - Tz\|^2 \\ &= \|x_k - Tz\|^2 - \|x_{k+1} - Tz\|^2 + 2\langle x_k - Tz, Tz - z \rangle + \|Tz - z\|^2 \end{aligned}$$

for every $k \in \mathbb{N}$. Summing these inequalities from $k = 1$ to n and dividing by n , we have

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^n \xi_k &\leq \frac{1}{n} (\|x_1 - Tz\|^2 - \|x_{n+1} - Tz\|^2) \\ &\quad + 2 \left\langle \frac{1}{n} \sum_{k=1}^n x_k - Tz, Tz - z \right\rangle + \|Tz - z\|^2 \\ &\leq \frac{1}{n} \|x_1 - Tz\|^2 + 2\langle z_n - Tz, Tz - z \rangle + \|Tz - z\|^2 \end{aligned}$$

for every $n \in \mathbb{N}$. Since z is a weak cluster point of $\{z_n\}$, there is a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that $z_{n_i} \rightharpoonup z$. Replacing n by n_i in the above inequality, we obtain

$$\frac{1}{n_i} \sum_{k=1}^{n_i} \xi_k \leq \frac{1}{n_i} \|x_1 - Tz\|^2 + 2\langle z_{n_i} - Tz, Tz - z \rangle + \|Tz - z\|^2.$$

Since $(1/n_i) \sum_{k=1}^{n_i} \xi_k \rightarrow 0$ and $z_{n_i} \rightarrow z$, we conclude that

$$0 \leq 2\langle z - Tz, Tz - z \rangle + \|Tz - z\|^2 = -\|Tz - z\|^2$$

and hence $Tz = z$. \square

Lemma 2.8 (Takahashi and Toyoda [12]). *Let F be a nonempty closed convex subset of a Hilbert space H , P the metric projection of H onto F , and $\{x_n\}$ a sequence in H such that $\|x_{n+1} - u\| \leq \|x_n - u\|$ for all $u \in F$ and $n \in \mathbb{N}$. Then $\{Px_n\}$ converges strongly to some point in F .*

3. FIXED POINT THEOREMS

In this section, we study existence of fixed points of λ -hybrid mappings.

The following theorem is a generalization of [1, Theorem 3.2] and [2, Theorem 4.1].

Theorem 3.1. *Let H be a Hilbert space, C a nonempty closed convex subset of H , $\{\lambda_n\}$ a sequence of real numbers such that $\lambda_n \rightarrow \lambda$, and $T_n: C \rightarrow C$ a λ_n -hybrid mapping for $n \in \mathbb{N}$. Let $\{x_n\}$ and $\{z_n\}$ be sequences in C defined by $x_1 \in C$,*

$$x_{n+1} = T_n x_n, \text{ and } z_n = \frac{1}{n} \sum_{k=1}^n x_k$$

for $n \in \mathbb{N}$. Suppose that $\{T_n\}$ is pointwise convergent and T denotes the pointwise limit of $\{T_n\}$, that is, $Tx = \lim_{n \rightarrow \infty} T_n x$ for $x \in C$. Then the following hold:

- (i) The mapping T is λ -hybrid and $\bigcap_{n=1}^{\infty} F(T_n) \subset F(T)$;
- (ii) if $\{x_n\}$ is bounded, then T has a fixed point and every weak cluster point of $\{z_n\}$ is a fixed point of T .

Proof. We first prove (1). Let $x, y \in C$ be fixed. Since each T_n is λ_n -hybrid, it follows that

$$\|T_n x - T_n y\|^2 \leq \|x - y\|^2 + 2(1 - \lambda_n)\langle x - T_n x, y - T_n y \rangle$$

for every $n \in \mathbb{N}$. Taking the limit $n \rightarrow \infty$, we have

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2(1 - \lambda)\langle x - Tx, y - Ty \rangle.$$

Thus T is λ -hybrid. Furthermore, let $u \in \bigcap_{n=1}^{\infty} F(T_n)$. Since T_n is pointwise convergent, $Tu = \lim_{n \rightarrow \infty} T_n u = u$ and hence $u \in F(T)$.

We next prove (2). Assume that $\{x_n\}$ is bounded. Then $\{z_n\}$ is also bounded and thus there exists a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that $z_{n_i} \rightarrow z \in C$. It is enough to show that z is a fixed point of T . Since T_n is λ_n -hybrid and $x_{n+1} = T_n x_n$, we have

$$\begin{aligned} \|x_{n+1} - Tz\|^2 &= \|x_{n+1} - T_n z + T_n z - Tz\|^2 \\ &= \|x_{n+1} - T_n z\|^2 + \|T_n z - Tz\|^2 + 2\langle x_{n+1} - T_n z, T_n z - Tz \rangle \\ &\leq \|x_n - z\|^2 + 2(1 - \lambda_n)\langle x_n - x_{n+1}, z - T_n z \rangle \\ &\quad + \|T_n z - Tz\|(\|T_n z - Tz\| + 2\|x_{n+1} - T_n z\|). \end{aligned}$$

Therefore, we conclude that

$$\mu_n + \epsilon_n \leq \|x_n - z\|^2 - \|x_{n+1} - Tz\|^2$$

for every $n \in \mathbb{N}$, where $\mu_n = 2(1 - \lambda_n)\langle x_{n+1} - x_n, z - T_n z \rangle$ and

$$\epsilon_n = -\|T_n z - Tz\|(\|T_n z - Tz\| + 2\|x_{n+1} - T_n z\|).$$

Since $\{x_n\}$ is bounded and both $\{\lambda_n\}$ and $\{T_n z\}$ are convergent, Lemma 2.6 shows that $(1/n) \sum_{k=1}^n \mu_k \rightarrow 0$, and hence $(1/n) \sum_{k=1}^n (\mu_k + \epsilon_k) \rightarrow 0$. Thus Lemma 2.7 implies that z is a fixed point of T . \square

A direct consequence of Theorem 3.1 is as follows:

Corollary 3.1. *Let H be a Hilbert space, C a nonempty closed convex subset of H , λ a real number, $T: C \rightarrow C$ a λ -hybrid mapping, and $\{\alpha_n\}$ a sequence in $[0, 1)$ such that $\alpha_n \rightarrow 0$. Let $\{x_n\}$ and $\{z_n\}$ be sequences in C defined by $x_1 \in C$,*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \text{ and } z_n = \frac{1}{n} \sum_{k=1}^n x_k$$

for $n \in \mathbb{N}$. Suppose that $\{x_n\}$ is bounded. Then T has a fixed point and every weak cluster point of $\{z_n\}$ is a fixed point of T .

Proof. Put $T_n = \alpha_n I + (1 - \alpha_n) T$ for $n \in \mathbb{N}$. Then Lemma 2.2 shows that $T_n: C \rightarrow C$ is $(\lambda - \alpha_n)/(1 - \alpha_n)$ -hybrid. It is clear that $(\lambda - \alpha_n)/(1 - \alpha_n) \rightarrow \lambda$ and T is the pointwise limit of $\{T_n\}$. Therefore, Theorem 3.1 implies the conclusion. \square

In particular, assuming that $\alpha_n = 0$ for each $n \in \mathbb{N}$ in Corollary 3.1, we obtain the following:

Corollary 3.2. ([2, Theorem 4.1]). *Let H, C, λ , and T be the same as in Corollary 3.1. Let x be a point in C and $\{z_n\}$ a sequence in C defined by*

$$z_n = \frac{1}{n} \sum_{k=1}^n T^{k-1} x$$

for $n \in \mathbb{N}$, where T^0 is the identity mapping on C . Suppose that $\{T^n x\}$ is bounded. Then T has a fixed point and every weak cluster point of $\{z_n\}$ is a fixed point of T .

4. MEAN CONVERGENCE THEOREMS

In this section, we prove some mean convergence theorems for a family of λ -hybrid mappings.

We first prove the following lemma, which is a variant of [2, Lemma 5.1].

Lemma 4.1. *Let H be a Hilbert space, C a nonempty closed convex subset of H , and $T_n: C \rightarrow C$ a quasi-nonexpansive mapping for $n \in \mathbb{N}$. Suppose that $\{T_n\}$ has a common fixed point. Let F be the set of common fixed points of $\{T_n\}$ and P the metric projection of H onto F . Let $\{x_n\}$ and $\{z_n\}$ be sequences in C defined by $x_1 \in C$,*

$$x_{n+1} = T_n x_n, \text{ and } z_n = \frac{1}{n} \sum_{k=1}^n x_k$$

for $n \in \mathbb{N}$. Then the following hold:

- (i) *The sequence $\{x_n\}$ is bounded and $\{P x_n\}$ converges strongly;*
- (ii) *if each weak cluster point of $\{z_n\}$ belongs to F , then $\{z_n\}$ converges weakly to the strong limit of $\{P x_n\}$.*

Proof. We first prove (1). Since T_n is quasi-nonexpansive,

$$\|x_{n+1} - u\| = \|T_n x_n - u\| \leq \|x_n - u\|$$

for all $u \in F$ and $n \in \mathbb{N}$. Thus $\{x_n\}$ is bounded and Lemma 2.8 implies that $\{P x_n\}$ converges strongly.

We next prove (2). Since $\{z_n\}$ is bounded by (1), there exists a weak cluster point z of $\{z_n\}$. Let $\{z_{n_i}\}$ be a subsequence of $\{z_n\}$ such that $z_{n_i} \rightharpoonup z$ and w the strong limit of $\{Px_n\}$. It is enough to show that $z = w$. Since P is the metric projection of H onto F and $z \in F$, it follows from (2.1) that

$$\langle z - Px_k, x_k - Px_k \rangle \leq 0$$

for every $k \in \mathbb{N}$. Since each T_k is quasi-nonexpansive and $Px_k \in F$, it follows from the definition of P that

$$\|x_{k+1} - Px_{k+1}\| \leq \|x_{k+1} - Px_k\| = \|T_k x_k - Px_k\| \leq \|x_k - Px_k\|$$

for every $k \in \mathbb{N}$. Therefore

$$\begin{aligned} \langle z - w, x_k - Px_k \rangle &= \langle z - Px_k, x_k - Px_k \rangle + \langle Px_k - w, x_k - Px_k \rangle \\ &\leq \langle Px_k - w, x_k - Px_k \rangle \\ &\leq \|Px_k - w\| \|x_k - Px_k\| \\ &\leq \|Px_k - w\| \|x_1 - Px_1\| \end{aligned}$$

for every $k \in \mathbb{N}$. Summing these inequalities from $k = 1$ to n_i and dividing by n_i , we have

$$\left\langle z - w, z_{n_i} - \frac{1}{n_i} \sum_{k=1}^{n_i} Px_k \right\rangle \leq \frac{1}{n_i} \sum_{k=1}^{n_i} \|Px_k - w\| \|x_1 - Px_1\|.$$

Since $z_{n_i} \rightarrow z$ as $i \rightarrow \infty$ and $Px_n \rightarrow w$ as $n \rightarrow \infty$, we obtain $\langle z - w, z - w \rangle \leq 0$ and hence $z = w$. This completes the proof. \square

Using Theorem 3.1 and Lemma 4.1, we obtain the following:

Theorem 4.1. *Let H be a Hilbert space, C a nonempty closed convex subset of H , $\{\lambda_n\}$ a sequence of real numbers such that $\lambda_n \rightarrow \lambda$, and $T_n: C \rightarrow C$ a λ_n -hybrid mapping for $n \in \mathbb{N}$. Let $\{x_n\}$ and $\{z_n\}$ be sequences in C defined by $x_1 \in C$,*

$$x_{n+1} = T_n x_n, \text{ and } z_n = \frac{1}{n} \sum_{k=1}^n x_k$$

for $n \in \mathbb{N}$. Suppose that $\{T_n\}$ is pointwise convergent, T denotes the pointwise limit of $\{T_n\}$, and $F(T) = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Then $\{z_n\}$ converges weakly to the strong limit of $\{Px_n\}$, where P is the metric projection of H onto $F(T)$.

Proof. Since T_n is λ_n -hybrid and $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$, each T_n is quasi-nonexpansive. Thus it follows from Lemma 4.1 that $\{x_n\}$ is bounded. Hence Theorem 3.1 shows that every weak cluster point of $\{z_n\}$ belongs to $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. Therefore, Lemma 4.1 implies the conclusion. \square

The following corollary is a direct consequence of Theorem 4.1.

Corollary 4.2. *Let H be a Hilbert space, C a nonempty closed convex subset of H , λ a real number, $T: C \rightarrow C$ a λ -hybrid mapping with a fixed point, and $\{\alpha_n\}$ a sequence in $[0, 1)$ such that $\alpha_n \rightarrow 0$. Let $\{x_n\}$ and $\{z_n\}$ be sequences in C defined by $x_1 \in C$,*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \text{ and } z_n = \frac{1}{n} \sum_{k=1}^n x_k$$

for $n \in \mathbb{N}$. Then $\{z_n\}$ converges weakly to the strong limit of $\{Px_n\}$, where P is the metric projection of H onto $F(T)$.

Proof. Put $T_n = \alpha_n I + (1 - \alpha_n)T$ for $n \in \mathbb{N}$. Then Lemma 2.2 shows that each $T_n: C \rightarrow C$ is $(\lambda - \alpha_n)/(1 - \alpha_n)$ -hybrid. It is clear that $(\lambda - \alpha_n)/(1 - \alpha_n) \rightarrow \lambda$ and T is the pointwise limit of $\{T_n\}$. It is also clear that $F(T_n) = F(T)$ for every $n \in \mathbb{N}$ and hence $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. Therefore, Theorem 4.1 implies the conclusion. \square

Using Corollary 4.2, we immediately obtain the following weak convergence theorem for a strictly pseudononspreading mapping, which is a generalization of [8, Theorem 3.1].

Corollary 4.3. (Osilike and Isiogugu [9, Theorem 3.1]) *Let $H, C, \{\alpha_n\}$, and P be the same as in Corollary 4.2. Let κ and β be real numbers with $0 \leq \kappa \leq \beta < 1$ and $T: C \rightarrow C$ a κ -strictly pseudononspreading mapping with a fixed point. Let $\{x_n\}$ and $\{z_n\}$ be sequences in C defined by $x_1 \in C$,*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)(\beta x_n + (1 - \beta)Tx_n), \text{ and } z_n = \frac{1}{n} \sum_{k=1}^n x_k$$

for $n \in \mathbb{N}$. Then $\{z_n\}$ converges weakly to the strong limit of $\{Px_n\}$.

Proof. Set $T_\beta = \beta I + (1 - \beta)T$. Then it follows from Lemma 2.5 that $-\beta/(1 - \beta)$ -hybrid. Obviously, $F(T) = F(T_\beta)$. Thus Corollary 4.2 implies the conclusion. \square

Assuming that $\alpha_n = 0$ for each $n \in \mathbb{N}$ in Corollary 4.2, we obtain the following:

Corollary 4.4. ([2, Theorem 5.2]) *Let H, C, λ, T , and P be the same as in Corollary 4.2. Let x be a point in C and $\{z_n\}$ a sequence in C defined by*

$$z_n = \frac{1}{n} \sum_{k=1}^n T^{k-1}x$$

for $n \in \mathbb{N}$, where T^0 is the identity mapping on C . Then $\{z_n\}$ converges weakly to the strong limit of $\{PT^n x\}$.

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