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OPTION PRICING FOR A JUMP DIFFUSION MODEL WITH FRACTIONAL STOCHASTIC VOLATILITY[◇]

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ABSTRACT. An alternative stochastic volatility model with jumps is proposed, in which stock prices follow a jump diffusion model and their stochastic volatility follows a fractional stochastic volatility model. By using an approximate method, we find a formulation for the European-style option in terms of the characteristic function of tail probabilities.

KEYWORDS : Fractional Brownian motion; Approximate approach; Stochastic Volatility; Jump diffusion model.
AMS Subject Classification: 60G22.

1. INTRODUCTION

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with filtration $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$. All processes that we shall consider in Section 1 and 2 will be defined in this space.

For $t \in [0, T]$ and $T < \infty$, a *geometric Brownian motion (gBm) model with jumps and with fractional stochastic volatility* is a model of the form

$$dS_t = S_t\left(\mu dt + \sqrt{v_t}dW_t\right) + S_{t-}Y_t dN_t, \tag{1.1}$$

where $\mu \in \Re$, $S = (S_t)_{t\in[0,T]}$ is a process representing a price of the underlying risky assets, $W = (W_t)_{t\in[0,T]}$ is the standard Brownian motion, $N = (N_t)_{t\in[0,T]}$ is a Poisson process with intensity λ, and $S_{t-}Y_t$ represents the amplitude of the jump which occurs at time t. We assume that the processes W and N are independent. The volatility process $v_t := \sigma_t^2$ in (1.1) is modeled by

$$dv_t = (\omega - \theta v_t)\, dt + \xi v_t dB_t, \tag{1.2}$$

where $\omega > 0$ is the mean long-term volatility, $\theta \in \Re$ is the rate at which the volatility reverts toward its long-term mean, $\xi > 0$ is the volatility of the volatility process,

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and $(B_t)_{t\in[0,T]}$ is a fractional Brownian motion. Assume that the proessess (S_t) and (v_t) are $\mathcal{F}_t$-measurable.

The notation S_{t-} means that whenever there is a jump, the value of the process before the jump is used on the left-hand side of the formula.

The fraction version of equation (1.1) is given by

$$dS_t = S_t\left(\mu dt + \sqrt{v_t}dB_t\right) + S_{t-}Y_t dN_t. \tag{1.3}$$

Recently, Intarasit and Sattayatham [1] showed that the process S_t in (1.3) can be approximated in $L_2(\Omega)$ by a semimartingale S_t^ε in the sense that $||S_t^\varepsilon - S_t||_{L_2(\Omega)} \to 0$ as $\varepsilon \to 0$, where S_t^ε satisfies the following equation

$$dS_t^\varepsilon = S_t^\varepsilon\left(\mu dt + \sqrt{v_t^\varepsilon}dW_t\right) + Y_t dN_t.$$

The purpose of this paper is to consider the problem of option pricing for the gBm model with jumps and with fractional stochastic volatility (1.1). But since the process S_t is a fractional process, we cannot apply Ito calculus directly. We shall thus work in another direction by finding a formula for option pricing for the process S_t^ε and using it as an approximation for pricing the model (1.1). In order to find such a formula, we shall work in the space of a risk-neutral probability measure. There are some authors who have investigated this problem before but not in the fractional case, for example Heston [2] and Yan and Hanson [3].

Recall that the fractional Brownian motion with *Hurst coeifficient* is a Gaussian process $B^H = (B_t^H)_{t\geq 0}$ with zero mean, and the covariance function is given by

$$R(t,s) = E\left[B_t^H B_s^H\right] = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right).$$

If $H = 1/2$, then $R(t,s) = min(t,s)$ and B_t^H is the usual standard Brownian motion. In the case $1/2 < H < 1$ the fractional Brownian motion exhibits statistical long-range dependency in the sense that $\rho_n := E[B_1^H\left(B_{n+1}^H - B_n^H\right)] > 0$ for all $n = 1, 2, 3, \ldots$ and $\sum_{n=1}^{\infty} \rho_n = \infty$ ([4], page 2). Hence, in financial modeling, one usually assumes that $H \in (1/2, 1)$. Put $\alpha = 1/2 - H$. It is known that a fractional Brownian motion B_t^H can be decomposed as follows:

$$B_t^H = \frac{1}{\Gamma(1+\alpha)}\left\{Z_t + \int_0^t (t-s)^{-\alpha} dW_s\right\}$$

where Γ is the gamma function, $Z_t = \int_{-\infty}^0 \left[(t-s)^{-\alpha} - (s)^{-\alpha}\right] dW_s$.

We suppose from now on that $0 < \alpha < 1/2$. The process Z_t has absolutely continuous trajectories, so it suffices to consider only the term

$$B_t = \int_0^t (t-s)^{-\alpha} dW_s, \tag{1.4}$$

that has a long-range dependence.

Note that B_t can be approximated by

$$B_t^{\varepsilon} = \int_0^t (t-s+\varepsilon)^{-\alpha} dW_s \tag{1.5}$$

in the sense that B_t^{ε} converges to B_t in $L_2(\Omega)$ as $\varepsilon \to 0$, uniform with respect to $t \in [0,T]$ (see [5]).

Since $(B_t^{\varepsilon})_{t\in[0,T]}$ is a continuous semimartingale then Itô calculus can be applied to the following stochastic differential equation (SDE)

$$dS_t^{\varepsilon} = S_t^{\varepsilon}(\mu dt + \sigma dB_t^{\varepsilon}),\ 0 \le t \le T.$$

Let S_t^{ε} be the solution of the above equation. Because of the convergence of B_t^{ε} to B_t in $L_2(\Omega)$ when $\varepsilon \to 0$, we shall define the *solution of a fractional stochastic differential equation* of the form

$$dS_t = S_t(\mu dt + \sigma dB_t),\ 0 \le t \le T,$$

to be a process S_t^* defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that the process S_t^{ε} converges to S_t^* in $L_2(\Omega)$ as $\varepsilon \to 0$ and the convergence is uniform with respect to $t \in [0,T]$. This definition will be applied to the other similar fractional stochastic differential equations which will appear later.

The rest of the paper is organized as follows. A risk-neutral for gBm model with a compound Poisson process and stochastic volatility model is described in section 2. The risk-neutral for gBm model with a compound Poisson process and fractional stochastic volatility model is also introduced in this section. The relationship between the stochastic differential equation and the partial differential equation for the jump diffusion process with stochastic volatility is presented in section 3. In section 4, an option price formula is given. Finally the closed-form solution for a European call option in terms of characteristic function is given in section 5.

2. RISK-NEUTRAL FOR A GBM WITH JUMPS

In this section, a risk-neutral for a gBm model combining jumps with stochastic volatility is introduced. Its solution will also be discussed in this section.

Firstly, let us rewrite the model (1.1) into an integral form as follows:

$$S_t = S_0 + \int_0^t \mu S_s ds + \int_0^t \sqrt{v_s} S_s dW_s + \int_0^t S_{s-} Y_s dN_s. \tag{2.1}$$

Note that the last term on the right hand side of equation (2.1) is defined by

$$\int_0^t S_{s-} Y_s dN_s = \sum_{n=1}^{N_t} \Delta S_n,$$

where

$$\Delta S_n := S_{T_n} - S_{T_n -} = S_{n-} Y_n.$$

The assumption $Y_n > 0$ always leads to positive values of the stock prices. The process $(Y_n)_{n\in\mathbb{N}}$ is assumed to be independently identically distributed (i.i.d.) with density $\phi_Y(y)$ and $(T_n)_{n\in\mathbb{N}}$ is a sequence of jump time.

In order to solve equation (2.1) with an initial condition $S_{t(t=0)} = S_0$, we assume that $E[\int_0^T v_s S_s^2 ds] < \infty$. Then, by an application of Itô's formula for the jump process ([6], Theorem 8.14, page 275) on equation (2.1) with $f(S_t, t) = log(S_t)$, we get

$$\log S_t = \log S_0 + \mu t - \frac{1}{2}\int_0^t v_s ds + \int_0^t \sqrt{v_s} dW_s + \int_0^t \log(1+Y_s)\, dN_s,$$

or, equivalently,

$$S_t = S_0 \exp\left(\mu t - \frac{1}{2}\int_0^t v_s ds + \int_0^t \sqrt{v_s} dW_s + \int_0^t \log(1+Y_s)\, dN_s\right).$$

It is assumed that a risk-neutral probability measure $\mathcal{M}$ exists; the asset price S_t, under this risk-neutral measure, follows a jump-diffusion process, with zero-mean, risk-free rate r, and stochastic variance v_t,

$$dS_t = S_t\left((r - \lambda E_{\mathcal{M}}[Y_t])\, dt + \sqrt{v_t} dW_t\right) + S_{t-} Y_t dN_t. \tag{2.2}$$

It is only necessary to know that the risk-neutral measure exists (see, [6] page 321). Hence, all processes to be discussed after this will be the processes under the risk-neutral probability measure $\mathcal{M}$.

Using an initial condition $S_{t(t=0)} = S_0 \in L_2(\Omega)$, its solution is given by

$$S_t = S_0 \exp\left(\int_0^t (r - \lambda E_{\mathcal{M}}(Y_s))\, ds - \frac{1}{2}\int_0^t v_s ds + \int_0^t \sqrt{v_s} dW_s + \int_0^t \log(1+Y_s)\, dN_s\right),$$

where v_t satisfies the following fractional SDE

$$dv_t = (\omega - \theta v_t)\, dt + \xi v_t dB_t, \tag{2.3}$$

with an initial condition $v_{t(t=0)} = v_0 \in L_2(\Omega)$.

For each $\varepsilon > 0$, consider an approximation model of equation (2.3);

$$dv_t^\varepsilon = (\omega - \theta v_t^\varepsilon)\, dt + \xi v_t^\varepsilon dB_t^\varepsilon. \tag{2.4}$$

By using the same initial condition as in equation (2.3), one can show that the solution v_t^ε of equation (2.4) converges in $L_2(\Omega)$ to the process

$$v_t = \left(v_0 + \omega \int_0^t \exp(\gamma s - \xi B_s)\, ds\right) \exp(\xi B_t - \gamma t)$$

for some real constant γ. ([1], Lemma 2). Hence, by definition, v_t is the solution of equation (2.3).

Now we consider an approximation model of equation (2.2);

$$dS_t^\varepsilon = S_t^\varepsilon\left((r - \lambda E_{\mathcal{M}}[Y_t])\, dt + \sqrt{v_t^\varepsilon} dW_t\right) + S_{t-}^\varepsilon Y_t dN_t, \tag{2.5}$$

and by using the same initial condition as in equation (2.2), we have

$$S_t^\varepsilon = S_0 \exp\left(\int_0^t (r - \lambda E_{\mathcal{M}}(Y_s))\, ds - \frac{1}{2}\int_0^t v_s^\varepsilon ds + \int_0^t \sqrt{v_s^\varepsilon} dW_s + \int_0^t \log(1+Y_s)\, dN_s\right). \tag{2.6}$$

Again, we can prove that ([1], Theorem 3) S_t^ε converges to S_t in $L_2(\Omega)$ as $\varepsilon \to 0$ and uniformly on $t \in [0, T]$.

3. PARTIAL INTEGRO-DIFFERENTIAL EQUATION FOR JUMP DIFFUSION MODEL WITH STOCHASTIC VOLATILITY

Consider the process $\overrightarrow{X}_t = \left(X_t^1, X_t^2\right)$ where X_t^1 and X_t^2 are processes in $\Re$ and satisfy the following equations:

$$\begin{aligned} dX_t^1 &= f_1(t)\,dt + g_1(t)\,dW_t + X_{t-}^1 Y_t dN_t, \\ dX_t^2 &= f_2(t)\,dt + g_2(t)\,d\overline{W}_t, \end{aligned} \tag{3.1}$$

where f_1, g_1, f_2, and g_2 are all continuous functions from $[0,T]$ into $\Re$.

Since every compound Poisson process can be represented as an integral form of Poisson random measure ([6], page 77) then the last term on the right hand side of equation (3.1) can be written as follows

$$\int_0^t X_{s-}^1 Y_s dN_s = \sum_{n=1}^{N_t} X_{n-}^1 Y_n = \sum_{n=1}^{N_t} \left[X_{T_n}^1 - X_{T_n-}^1\right] = \int_0^t \int_{\Re} X_{s-}^1 z J_Z(ds\ dz)$$

where Y_n are i.i.d random variables with density $\phi_Y(y)$ and J_Z is a Poisson random measure of the process $Z_t = \sum_{n=1}^{N_t} Y_n$ with intensity measure $\lambda\phi_Y(dz)dt$.

Let $U(\overrightarrow{x})$ be a bounded real function on $\Re^2$ and twice continuously differentiable in $\overrightarrow{x} = (x_1, x_2) \in \Re^2$ and

$$u(\overrightarrow{x}, t) = E\left[U\left(\overrightarrow{X_T}\right) | \overrightarrow{X}_t = \overrightarrow{x}\right]. \tag{3.2}$$

By the two dimensional Dynkin's formula ([7], Theorem 7.7, page 203), u is a solution of the partial integro-differential equation (PIDE)

$$0 = \frac{\partial u(\overrightarrow{x}, t)}{\partial t} + \mathcal{A}u(\overrightarrow{x}, t) + \lambda \int_{\Re} \left[u(\overrightarrow{x} + \overrightarrow{y}, t) - u(\overrightarrow{x}, t)\right] \phi_Y(y)dy,$$

subject to the final condition $u(\overrightarrow{x}, T) = U(\overrightarrow{x})$ where $\overrightarrow{y} = (y, 0) \in \Re^2$. The notation $\mathcal{A}$ is defined by

$$\begin{aligned} \mathcal{A}u(\overrightarrow{x}, t) &= f_1(t)\frac{\partial u(\overrightarrow{x}, t)}{\partial x_1} + f_2(t)\frac{\partial u(\overrightarrow{x}, t)}{\partial x_2} \\ &\quad + \frac{1}{2}g_1^2(t)\frac{\partial^2 u(\overrightarrow{x}, t)}{\partial x_1^2} + \rho g_1(t)g_2(t)\frac{\partial^2 u(\overrightarrow{x}, t)}{\partial x_1 \partial x_2} + \frac{1}{2}g_2^2(t)\frac{\partial^2 u(\overrightarrow{x}, t)}{\partial x_2^2}, \end{aligned}$$

and the correlation ρ defined by $\rho = Corr\left[dW_t, d\overline{W}_t\right]$.

4. PRICING A EUROPEAN CALL OPTION

Let C denote the price at time t of a *European style call option* on the current price of the underlying asset S_t with strike price K and expiration time T.

The *terminal payoff of a European call option* on the underling stock S with strike price K is

$$\max\left(S_T - K, 0\right).$$

This means that the holder will exercise his right only if $S_T > K$ and then his gain is $S_T - K$. Otherwise, if $S_T \le K$, then the holder will buy the underlying asset from the market and the value of the option is zero.

Assuming the risk-free interest rate r is constant over the lifetime of the option, the price of the European call at time t is equal to the discounted conditional expected payoff

$$C\left(S_t, v_t, t; K, T\right)$$

$$
\begin{aligned}
&= e^{-r(T-t)} E_{\mathcal{M}} \left[\max\left(S_T - K, 0\right) \mid S_t, v_t\right] \\
&= e^{-r(T-t)} \left(\int_K^{\infty} S_T P_{\mathcal{M}} \left(S_T \mid S_t, v_t\right) dS_T - K \int_K^{\infty} P_{\mathcal{M}} \left(S_T \mid S_t, v_t\right) dS_T \right) \\
&= S_t \left(\frac{1}{e^{r(T-t)} S_t} \int_K^{\infty} S_T P_{\mathcal{M}} \left(S_T \mid S_t, v_t\right) dS_T \right) - K e^{-r(T-t)} \int_K^{\infty} P_{\mathcal{M}} \left(S_T \mid S_t, v_t\right) dS_T \\
&= S_t \left(\frac{1}{E_{\mathcal{M}} \left[S_T \mid S_t, v_t\right]} \int_K^{\infty} S_T P_{\mathcal{M}} \left(S_T \mid S_t, v_t\right) dS_T \right) \\
&\quad - K e^{-r(T-t)} \int_K^{\infty} P_{\mathcal{M}} \left(S_T \mid S_t, v_t\right) dS_T \\
&= S_t P_1 \left(S_t, v_t, t; K, T\right) - K e^{-r(T-t)} P_2 \left(S_t, v_t, t; K, T\right) \qquad (4.1)
\end{aligned}
$$

where $E_{\mathcal{M}}$ is the expectation with respect to the risk-neutral probability measure, $P_{\mathcal{M}} \left(S_T \mid S_t, v_t\right)$ is the corresponding conditional density given $\left(S_t, v_t\right)$, and

$$
P_1 \left(S_t, v_t, t; K, T\right) = \left(\int_K^{\infty} S_T P_{\mathcal{M}} \left(S_T \mid S_t, v_t\right) dS_T \right) / E_{\mathcal{M}} \left[S_T \mid S_t, v_t\right].
$$

Note that P_1 is the risk-neutral probability that $S_T > K$ (since the integrand is nonnegative and the integral over $[0, \infty)$ is one), and finally, that

$$
P_2 \left(S_t, v_t, t; K, T\right) = \int_K^{\infty} P_{\mathcal{M}} \left(S_T \mid S_t, v_t\right) dS_T = \Pr\mathrm{ob} \left(S_T > K \mid S_t, v_t\right)
$$

is the risk-neutral in-the-money probability. Moreover, $E_{\mathcal{M}} \left[S_T \mid S_t, v_t\right] = e^{r(T-t)} S_t$ for $t \geq 0$.

Note that we do not have a closed form solution for these probabilities. However, these probabilities are related to characteristic functions which have closed form solutions as will be seen in Lemma 2.

We would like to compute the price of a European call option with strike price K and maturity T of the model (2.2) for which its fractional stochastic volatility satisfies equation (2.3).

To do this, consider the logarithm of S_t^{ε}, namely L_t^{ε}, i.e. $L_t^{\varepsilon} = \log\left(S_t^{\varepsilon}\right)$ where S_t^{ε} satisfies equation (2.6) and its inverse $S_t^{\varepsilon} = \exp\left(L_t^{\varepsilon}\right)$. Denote $\kappa = \log\left(K\right)$ the logarithm of the strike price.

We now refer to equation (2.4), since this approximate model is driven by a semimartingale B_t^{ε} and hence there is no opportunity of arbitrage. This is the advantage of our approximate approach and we will use this model for pricing the European call option instead of (2.3).

Note that we can write

$$
dB_t^{\varepsilon} = \alpha \varphi_t^{\varepsilon} dt + \varepsilon^{\alpha} dW_t \qquad (4.2)
$$

where $\varphi_t^{\varepsilon} = \int_0^t (t - u + \varepsilon)^{1-\alpha} dW_u$, $\alpha = 1/2 - H$ and $0 < \alpha < 1/2$ ([5], Lemma 2.1).

Substituting (4.2) into equation (2.4), we obtain

$$
dv_t^{\varepsilon} = \left(\omega + \left(\alpha \xi \varphi_t^{\varepsilon} - \theta\right) v_t^{\varepsilon}\right) dt + \xi \varepsilon^{\alpha} v_t^{\varepsilon} dW_t. \qquad (4.3)
$$

Consider the SDE (2.2) and (4.3). Define a function U on $\Re^2$ as follows:

$$U(x_1, x_2) = e^{-r(T-t)} max(e^{x_1} - \kappa, 0).$$

By virtue of equation (3.2),

$$\begin{aligned} u(\overrightarrow{x}, t) &= E\left[U\left(\overrightarrow{X_T}\right) | \overrightarrow{X}_t = \overrightarrow{x}\right] \\ &= e^{-r(T-t)} E_{\mathcal{M}}\left[\max\left(\exp\left(L_t^\varepsilon\right) - \kappa, 0\right) \mid L_t^\varepsilon = \ell^\varepsilon, v_t^\varepsilon = v^\varepsilon\right] \\ &:= C\left(\ell^\varepsilon, v^\varepsilon, t; \kappa, T\right) \end{aligned}$$

satisfies the following PIDE:

$$\begin{aligned} 0 \quad = \quad & \frac{\partial C}{\partial t} + f_1 \frac{\partial C}{\partial \ell^\varepsilon} + f_2 \frac{\partial C}{\partial v^\varepsilon} + \frac{1}{2} g_1^2 \frac{\partial^2 C}{\partial (\ell^\varepsilon)^2} + \rho g_1 g_2 \frac{\partial^2 C}{\partial \ell^\varepsilon \partial v^\varepsilon} + \frac{1}{2} g_2^2 \frac{\partial^2 C}{\partial (v^\varepsilon)^2} \\ & -rC + \lambda \int\limits_{\Re} \left[C\left(\ell^\varepsilon + y, v^\varepsilon, t; \kappa, T\right) - C\left(\ell^\varepsilon, v^\varepsilon, t; \kappa, T\right)\right] \phi_Y(y) dy. \end{aligned} \tag{4.4}$$

In the current state variable, the last line of equation (4.1) becomes

$$C\left(\ell^\varepsilon, v^\varepsilon, t; \kappa, T\right) = e^{\ell^\varepsilon} P_1\left(\ell^\varepsilon, v^\varepsilon, t; \kappa, T\right) - e^{\kappa - r(T-t)} P_2\left(\ell^\varepsilon, v^\varepsilon, t; \kappa, T\right). \tag{4.5}$$

The following lemma shows the relationship between P_1 and P_2 in the option value of the equation (4.5).

Lemma 4.1. *The functions P_1 and P_2 in the option value of the equation (4.5) satisfy the following PIDEs*

$$\begin{aligned} 0 = & \frac{\partial P_1}{\partial t} + A\left[P_1\right]\left(\ell^\varepsilon, v^\varepsilon, t; \kappa, T\right) + v^\varepsilon \frac{\partial P_1}{\partial \ell^\varepsilon} + \rho \xi \varepsilon^\alpha (v^\varepsilon)^{3/2} \frac{\partial P_1}{\partial v^\varepsilon} \\ & + \left(r - \lambda E_{\mathcal{M}}\left(Y_t\right)\right) P_1 + \lambda \int\limits_{\Re} \left[\left(e^y - 1\right) P_1\left(\ell^\varepsilon + y, v^\varepsilon, t; \kappa, T\right)\right] \phi_Y(y) dy, \end{aligned}$$

subject to the boundary condition at expiration time $t = T$;

$$P_1\left(\ell^\varepsilon, v^\varepsilon, T; \kappa, T\right) = 1_{\ell^\varepsilon > \kappa}. \tag{4.6}$$

Moreover, P_2 satisfies the equation

$$0 = \frac{\partial P_2}{\partial t} + A\left[P_2\right]\left(\ell^\varepsilon, v^\varepsilon, t; \kappa, T\right) + rP_2,$$

subject to the boundary condition at expiration time $t = T$;

$$P_2\left(\ell^\varepsilon, v^\varepsilon, T; \kappa, T\right) = 1_{\ell^\varepsilon > \kappa}, \tag{4.7}$$

where

$$\begin{aligned} A[f]\left(\ell^\varepsilon, v^\varepsilon, t; \kappa, T\right) := & \left(r - \lambda E\left(Y_t\right) - \frac{1}{2} v^\varepsilon\right) \frac{\partial f}{\partial \ell^\varepsilon} + \left(\omega + \left(\alpha \xi \varphi_t^\varepsilon - \theta\right) v^\varepsilon\right) \frac{\partial f}{\partial v^\varepsilon} \\ & + \frac{1}{2} v^\varepsilon \frac{\partial^2 f}{(\partial \ell^\varepsilon)^2} + \rho \xi \varepsilon^\alpha (v^\varepsilon)^{3/2} \frac{\partial^2 f}{\partial \ell^\varepsilon \partial v^\varepsilon} + \frac{1}{2} \xi^2 \varepsilon^{2\alpha} (v^\varepsilon)^2 \frac{\partial^2 f}{\partial (v^\varepsilon)^2} \\ & - rf + \lambda \int\limits_{\Re} \left[f\left(\ell^\varepsilon + y, v^\varepsilon, t; \kappa, T\right) - f\left(\ell^\varepsilon, v^\varepsilon, t; \kappa, T\right)\right] \phi_Y(y) dy. \end{aligned} \tag{4.8}$$

Note that $1_{\ell^\varepsilon > \kappa} = 1$ if $\ell^\varepsilon > \kappa$ and otherwise $1_{\ell^\varepsilon > \kappa} = 0$.

Proof. We plan to substitute equation (4.5) into equation (4.4). Firstly, we compute

$$\begin{aligned}
\frac{\partial C}{\partial t} &= e^{\ell^\varepsilon}\frac{\partial P_1}{\partial t} - e^{\kappa - r(T-t)}\frac{\partial P_2}{\partial t} - re^{\kappa - r(T-t)}P_2 \\
\frac{\partial C}{\partial \ell^\varepsilon} &= e^{\ell^\varepsilon}\frac{\partial P_1}{\partial \ell^\varepsilon} + e^{\ell^\varepsilon}P_1 - e^{\kappa - r(T-t)}\frac{\partial P_2}{\partial \ell^\varepsilon} \\
\frac{\partial C}{\partial v^\varepsilon} &= e^{\ell^\varepsilon}\frac{\partial P_1}{\partial v^\varepsilon} - e^{\kappa - r(T-t)}\frac{\partial P_2}{\partial v^\varepsilon} \\
\frac{\partial^2 C}{(\partial \ell^\varepsilon)^2} &= e^{\ell^\varepsilon}\frac{\partial^2 P_1}{(\partial \ell^\varepsilon)^2} + 2e^{\ell^\varepsilon}\frac{\partial P_1}{\partial \ell^\varepsilon} + P_1 e^{\ell^{\varepsilon\varepsilon}} - e^{\kappa - r(T-t)}\frac{\partial^2 P_2}{(\partial \ell^\varepsilon)^2} \\
\frac{\partial^2 C}{\partial \ell^\varepsilon \partial v^\varepsilon} &= e^{\ell^\varepsilon}\frac{\partial^2 P_1}{\partial \ell^\varepsilon \partial v^\varepsilon} + e^{\ell^\varepsilon}\frac{\partial P_1}{\partial v^\varepsilon} - e^{\kappa - r(T-t)}\frac{\partial^2 P_2}{\partial \ell^\varepsilon \partial v^\varepsilon} \\
\frac{\partial^2 C}{\partial (v^\varepsilon)^2} &= e^{\ell^\varepsilon}\frac{\partial^2 P_1}{\partial (v^\varepsilon)^2} - e^{\kappa - r(T-t)}\frac{\partial^2 P_2}{\partial (v^\varepsilon)^2}
\end{aligned}$$

and

$$\begin{aligned}
& C\left(\ell^\varepsilon + y, v^\varepsilon, t; \kappa, T\right) - C\left(\ell^\varepsilon, v^\varepsilon, t; \kappa, T\right) \\
= & \left[e^{(\ell^\varepsilon + y)}P_1\left(\ell^\varepsilon + y, v^\varepsilon, t; \kappa, T\right) - e^{\kappa - r(T-t)}P_2\left(\ell^\varepsilon + y, v^\varepsilon, t; \kappa, T\right)\right] \\
& - \left[e^{\ell^\varepsilon}P_1\left(\ell^\varepsilon, v^\varepsilon, t; \kappa, T\right) - e^{\kappa - r(T-t)}P_2\left(\ell^\varepsilon, v^\varepsilon, t; \kappa, T\right)\right] \\
= & \left[\begin{array}{c} e^{\ell^\varepsilon}\left(e^y P_1\left(\ell^\varepsilon + y, v^\varepsilon, t; \kappa, T\right) - P_1\left(\ell^\varepsilon + y, v^\varepsilon, t; \kappa, T\right)\right) \\ + \left(e^{\ell^\varepsilon}P_1\left(\ell^\varepsilon + y, v^\varepsilon, t; \kappa, T\right) - e^{\ell^\varepsilon}P_1\left(\ell^\varepsilon, v^\varepsilon, t; \kappa, T\right)\right) \end{array}\right] \\
& - e^{\kappa - r(T-t)}\left[P_2\left(\ell^\varepsilon + y, v^\varepsilon, t; \kappa, T\right) - P_2\left(\ell^\varepsilon, v^\varepsilon, t; \kappa, T\right)\right] \\
= & e^{\ell^\varepsilon}\left(e^y - 1\right)P_1\left(\ell^\varepsilon + y, v^\varepsilon, t; \kappa, T\right) + e^{\ell^\varepsilon}\left(P_1\left(\ell^\varepsilon + y, v^\varepsilon, t; \kappa, T\right) - P_1\left(\ell^\varepsilon, v^\varepsilon, t; \kappa, T\right)\right) \\
& - e^{\kappa - r(T-t)}\left[P_2\left(\ell^\varepsilon + y, v^\varepsilon, t; \kappa, T\right) - P_2\left(\ell^\varepsilon, v^\varepsilon, t; \kappa, T\right)\right].
\end{aligned}$$

Substitute all terms above in equation (4.4) and separate it by assumed independent terms P_1 and P_2. This gives two PIDEs for the risk-neutralized probability $P_j\left(\ell^\varepsilon, v^\varepsilon, t; \kappa, T\right)$, $j = 1, 2$:

$$\begin{aligned}
0 = & \frac{\partial P_1}{\partial t} + \left(r - \lambda E_{\mathcal{M}}\left(Y_t\right) - \frac{1}{2}v^\varepsilon\right)\left(\frac{\partial P_1}{\partial \ell^\varepsilon} + P_1\right) + \left(\omega + \left(\alpha\xi\varphi_t^\varepsilon - \theta\right)v^\varepsilon\right)\frac{\partial P_1}{\partial v^\varepsilon} \\
& + \frac{1}{2}v^\varepsilon\left(\frac{\partial^2 P_1}{(\ell^\varepsilon)^2} + 2\frac{\partial P_1}{\partial \ell^\varepsilon} + P_1\right) + \rho\xi\varepsilon^\alpha (v^\varepsilon)^{3/2}\left(\frac{\partial^2 P_1}{\partial \ell^\varepsilon \partial v^\varepsilon} + \frac{\partial P_1}{\partial v^\varepsilon}\right) + \frac{1}{2}\xi^2\varepsilon^{2\alpha}(v^\varepsilon)^2\frac{\partial^2 P_1}{\partial (v^\varepsilon)^2} \\
& - rP_1 + \lambda\int_{\Re}\left[\begin{array}{c} \left(e^y - 1\right)P_1\left(\ell^\varepsilon + y, v^\varepsilon, t; \kappa, T\right) \\ + \left(P_1\left(\ell^\varepsilon + y, v^\varepsilon, t; \kappa, T\right) - P_1\left(\ell^\varepsilon, v^\varepsilon, t; \kappa, T\right)\right) \end{array}\right]\phi_Y(y)dy \qquad (4.9)
\end{aligned}$$

subject to the boundary condition at the expiration time $t = T$ according to equation (4.6).

By using the notation in equation (4.8), PIDE (4.9) becomes

$$\begin{aligned}
0 = & \frac{\partial P_1}{\partial t} + \left(\begin{array}{c} A\left[P_1\right]\left(\ell^\varepsilon, v^\varepsilon, t; \kappa, T\right) + v^\varepsilon\frac{\partial P_1}{\partial \ell^\varepsilon} + \rho\xi\varepsilon^\alpha (v^\varepsilon)^{3/2}\frac{\partial P_1}{\partial v^\varepsilon} + \left(r - \lambda E_{\mathcal{M}}\left(Y_t\right)\right)P_1 \\ + \lambda\int_{\Re}\left[\left(e^y - 1\right)P_1\left(\ell^\varepsilon + y, v^\varepsilon, t; \kappa, T\right)\right]\phi_Y(y)dy \end{array}\right) \\
:= & \frac{\partial P_1}{\partial t} + A_1\left[P_1\right]\left(\ell^\varepsilon, v^\varepsilon, t; \kappa, T\right).
\end{aligned}$$

For $P_2\left(\ell^\varepsilon, v^\varepsilon, t; \kappa, T\right)$:

$$0 = \frac{\partial P_2}{\partial t} + rP_2 + \left(r - \lambda E_{\mathcal{M}}\left(Y_t\right) - \frac{1}{2}v^\varepsilon\right)\left(\frac{\partial P_2}{\partial \ell^\varepsilon}\right) + \left(\omega + \left(\alpha\xi\varphi_t^\varepsilon - \theta\right)v^\varepsilon\right)\frac{\partial P_2}{\partial v^\varepsilon}$$

$$+\frac{1}{2}v^{\varepsilon}\frac{\partial^2 P_2}{\partial(\ell^{\varepsilon})^2}+\rho\xi\varepsilon^{\alpha}(v^{\varepsilon})^{3/2}\frac{\partial^2 P_1}{\partial\ell^{\varepsilon}\partial v^{\varepsilon}}+\frac{1}{2}\xi^2\varepsilon^{2\alpha}(v^{\varepsilon})^2\frac{\partial^2 P_1}{\partial(v^{\varepsilon})^2}$$
$$-rP_2+\lambda\int_{\Re}\left[P_2\left(\ell^{\varepsilon}+y,v^{\varepsilon},t;\kappa,T\right)-P_2\left(\ell^{\varepsilon},v^{\varepsilon},t;\kappa,T\right)\right]\phi_Y(y)dy \tag{4.10}$$

subject to the boundary condition at expiration time $t=T$ according to equation (4.7).

Again, by using the notation (4.8), PIDE (4.10) becomes

$$\begin{aligned}0&=\frac{\partial P_2}{\partial t}+\left(A\left[P_2\right]\left(\ell^{\varepsilon},v^{\varepsilon},t;\kappa,T\right)+rP_2\right)\\&:=\frac{\partial P_2}{\partial t}+A_2\left[P_2\right]\left(\ell^{\varepsilon},v^{\varepsilon},t;\kappa,T\right)\end{aligned}$$

The proof is now completed. □

5. A CLOSED-FORM SOLUTION FOR EUROPEAN CALL OPTIONS

For $j=1,2,$ the characteristic functions for $P_j\left(\ell^{\varepsilon},v^{\varepsilon},t;\kappa,T\right)$, with respect to the variable κ are defined by

$$f_j\left(\ell^{\varepsilon},v^{\varepsilon},t;x,T\right):=-\int_{-\infty}^{\infty}e^{ix\kappa}dP_j\left(\ell^{\varepsilon},v^{\varepsilon},t;\kappa,T\right),$$

with a minus sign to account for the negativity of the measure dP_j. Note that f_j also satisfies similar PIDEs

$$\frac{\partial f_j}{\partial t}+A_j[f_j]\left(\ell^{\varepsilon},v^{\varepsilon},t;\kappa,T\right)=0, \tag{5.1}$$

with the respective boundary conditions

$$f_j\left(\ell^{\varepsilon},v^{\varepsilon},T;x,T\right)=-\int_{-\infty}^{\infty}e^{ix\kappa}dP_j\left(\ell^{\varepsilon},v^{\varepsilon},T;\kappa,T\right)=-\int_{-\infty}^{\infty}e^{ix\kappa}(-\delta(\ell^{\varepsilon}-\kappa)d\kappa)=e^{ix\ell^{\varepsilon}},$$

since

$$dP_j\left(\ell^{\varepsilon},v^{\varepsilon},T;\kappa,T\right)=d1_{\ell^{\varepsilon}>\kappa}=dH(\ell^{\varepsilon}-\kappa)=-\delta(\ell^{\varepsilon}-\kappa)d\kappa.$$

The following lemma shows how to calculate the functions P_1 and P_2 as they appeared in Lemma 1.

Lemma 5.1. *The functions P_1 and P_2 can be calculated by the inverse Fourier transforms of the characteristic function, i.e.*

$$P_j\left(\ell^{\varepsilon},v^{\varepsilon},t;\kappa,T\right)=\frac{1}{2}+\frac{1}{\pi}\int_{0+}^{+\infty}Re\left[\frac{e^{-ix\kappa}f_j\left(\ell^{\varepsilon},v^{\varepsilon},t;x,T\right)}{ix}\right]dx,$$

for $j=1,2$, with $Re[\cdot]$ denoting the real component of a complex number.

By letting $\tau=T-t$. (i) The characteristic function f_1 is given by

$$f_1\left(\ell^{\varepsilon},v^{\varepsilon},t;x,t+\tau\right)=\exp\left(g_1\left(\tau\right)+v^{\varepsilon}h_1\left(\tau\right)+ix\ell^{\varepsilon}\right),$$

where

$$g_1\left(\tau\right)\quad=\quad\left[\left(r-\lambda E\left(Y_t\right)\right)ix-\lambda E_{\mathcal{M}}\left(Y_t\right)\right]\tau+\tau\lambda\int_{\Re}\left(e^{(ix+1)y}-1\right)\phi_Y(y)dy$$

$$-\frac{2\omega}{\xi^2\varepsilon^{2\alpha}v^\varepsilon}\left[\log\left(1-\frac{(\Delta_1+\eta_1)+\left(1-e^{\Delta_1\tau}\right)}{2\Delta_1}\right)+(\Delta_1+\eta_1)\,\tau\right],$$

$$h_1(\tau)=\frac{\left(\eta_1^2-\Delta_1^2\right)\left(e^{\Delta_1\tau}-1\right)}{\xi^2\varepsilon^{2\alpha}v^\varepsilon\left(\eta_1+\Delta_1-(\eta_1-\Delta_1)e^{\Delta_1\tau}\right)},$$

$$\eta_1=\rho\xi\varepsilon^\alpha\sqrt{v^\varepsilon}\,(1+ix)+(\alpha\xi\varphi_t^\varepsilon-\theta)\,,$$

and

$$\Delta_1=\sqrt{\eta_1^2-\xi^2\varepsilon^{2\alpha}v^\varepsilon ix\,(ix+1)}.$$

(ii) The characteristic function f_2 is given by

$$f_2\left(\ell^\varepsilon,v^\varepsilon,t;x,t+\tau\right)=\exp\left(g_2(\tau)+v^\varepsilon h_2(\tau)+ix\ell^\varepsilon+r\tau\right),$$

where

$$\begin{aligned} g_2(\tau) &= \left[(r-\lambda E_{\mathcal{M}}(Y_t))\,iy-r\right]\tau+\tau\lambda\int_{\Re}\left(e^{ixy}-1\right)\phi_Y(y)dy \\ &\quad -\frac{2\omega}{\xi^2\varepsilon^{2\alpha}v^\varepsilon}\left[\log\left(1-\frac{(\Delta_2+\eta_2)+\left(1-e^{\Delta_2\tau}\right)}{2\Delta_1}\right)+(\Delta_2+\eta_2)\,\tau\right],\end{aligned}$$

$$h_2(\tau)=\frac{\left(\eta_2^2-\Delta_2^2\right)\left(e^{\Delta_2\tau}-1\right)}{\xi^2\varepsilon^{2\alpha}v^\varepsilon\left(\eta_2+\Delta_2-(\eta_2-\Delta_2)e^{\Delta_2\tau}\right)},$$

$$\eta_2=\rho\xi\varepsilon^\alpha\sqrt{v^\varepsilon}ix+(\alpha\xi\varphi_t^\varepsilon-\theta)\,,$$

and

$$\Delta_2=\sqrt{\eta_2^2+\xi^2\varepsilon^{2\alpha}v^\varepsilon ix\,(ix-1)}.$$

Proof. Proof of (i). To solve for the characteristic explicitly, letting $\tau=T-t$ be the time-to-go, we conjecture that the function f_1 is given by

$$f_1\left(\ell^\varepsilon,v^\varepsilon,t;x,t+\tau\right)=\exp\left(g_1(\tau)+v^\varepsilon h_1(\tau)+ix\ell^\varepsilon\right), \tag{5.2}$$

and the boundary condition $g_1(0)=0=h_1(0)$. This conjecture exploits the linearity of the coefficient in PIDE (5.1).

Note that the characteristic function of f_1 always exists. In order to substitute (5.2) into (5.1), firstly, we compute

$$\frac{\partial f_1}{\partial t}=\left(-g_1'(\tau)-v^\varepsilon h_1'(\tau)\right)f_1,\ \frac{\partial f_1}{\partial\ell^\varepsilon}=ixf_1,\ \frac{\partial f_1}{\partial v^\varepsilon}=h_1(\tau)f_1,$$

$$\frac{\partial^2 f_1}{(\partial\ell^\varepsilon)^2}=-x^2f_1,\ \frac{\partial^2 f_1}{\partial\ell^\varepsilon\partial v^\varepsilon}=ixh_1(\tau)f_1,\ \frac{\partial^2 f_1}{\partial(v^\varepsilon)^2}=h_1^2(\tau)f_1,$$

$$f_1\left(\ell^\varepsilon+y,v^\varepsilon,t;x,t+\tau\right)-f_1\left(\ell^\varepsilon,v^\varepsilon,t;x,t+\tau\right)=\left(e^{ixy}-1\right)f_1\left(\ell^\varepsilon,v^\varepsilon,t;x,t+\tau\right),$$

and

$$\begin{aligned}\left(e^y-1\right)f_1\left(\ell^\varepsilon+y,v^\varepsilon,t;x,t+\tau\right) &= \left(e^y-1\right)e^{g_1(\tau)+v^\varepsilon h_1(\tau)+ix(\ell^\varepsilon+y)} \\ &= \left(e^y-1\right)e^{ixy}f_1\left(\ell^\varepsilon,v^\varepsilon,t;x,t+\tau\right).\end{aligned}$$

Substituting all the above terms into equation (5.1) and after canceling the common factor of f_1, we get a simplified form as follows:

$$\begin{aligned} 0 &= -g_1'(\tau)-v^\varepsilon h_1'(\tau)+\left(r-\lambda E_{\mathcal{M}}(Y_t)+\frac{1}{2}v^\varepsilon\right)ix \\ &\quad +\left(\left(\omega+\left(\alpha\xi\varphi_t^\varepsilon-\theta\right)v^\varepsilon\right)+\rho\xi\varepsilon^\alpha(v^\varepsilon)^{3/2}\right)h_1(\tau)\end{aligned}$$

$$-\frac{1}{2}v^{\varepsilon}x^2 + \rho\xi\varepsilon^{\alpha}(v^{\varepsilon})^{3/2}ixh_1(\tau) + \frac{1}{2}\xi^2\varepsilon^{2\alpha}(v^{\varepsilon})^2h_1^2(\tau)$$
$$-\lambda E_{\mathcal{M}}(Y_t) + \lambda \int_{\Re} \left(e^{(ix+1)y} - 1\right)\phi_Y(y)dy.$$

By separating the order v^{ε} and ordering the remaining terms, we can reduce it to two ordinary differential equations (ODEs),

$$h_1'(\tau) = \frac{1}{2}\xi^2\varepsilon^{2\alpha}v^{\varepsilon}h_1^2(\tau) + \left(\rho\xi\varepsilon^{\alpha}\sqrt{v^{\varepsilon}}(1+ix) + (\alpha\xi\varphi_t^{\varepsilon} - \theta)\right)h_1(\tau) + \frac{1}{2}ix - \frac{1}{2}x^2, \tag{5.3}$$

$$g_1'(\tau) = \omega h_1(\tau) + (r - \lambda E_{\mathcal{M}}(Y_t))\,ix - \lambda E_{\mathcal{M}}(Y_t) + \lambda\int_{\Re}\left(e^{(ix+1)y} - 1\right)\phi_Y(y)dy. \tag{5.4}$$

Let $\eta_1 = \rho\xi\varepsilon^{\alpha}\sqrt{v^{\varepsilon}}(1+ix) + (\alpha\xi\varphi_t^{\varepsilon} - \theta)$ and substitute it to equation (5.3). We get

$$\begin{aligned} h_1'(\tau) &= \frac{1}{2}\xi^2\varepsilon^{2\alpha}v^{\varepsilon}\left(h_1^2(\tau) + \frac{2\eta_1}{\xi^2\varepsilon^{2\alpha}v^{\varepsilon}}h_1(\tau) + \frac{1}{\xi^2\varepsilon^{2\alpha}v^{\varepsilon}}ix(ix+1)\right) \\ &= \frac{1}{2}\xi^2\varepsilon^{2\alpha}\left(h_1(\tau) + \frac{2\eta_1 + \sqrt{4\eta_1^2 - 4\xi^2\varepsilon^{2\alpha}v^{\varepsilon}ix(ix+1)}}{2\xi^2\varepsilon^{2\alpha}v^{\varepsilon}}\right) \\ &\quad \times\left(h_1(\tau) + \frac{2\eta_1 - \sqrt{4\eta_1^2 - 4\xi^2\varepsilon^{2\alpha}v^{\varepsilon}ix(ix+1)}}{2\xi^2\varepsilon^{2\alpha}v^{\varepsilon}}\right) \\ &= \frac{1}{2}\xi^2\varepsilon^{2\alpha}v^{\varepsilon}\left(h_1(\tau) + \frac{\eta_1+\Delta_1}{\xi^2\varepsilon^{2\alpha}v^{\varepsilon}}\right)\left(h_1(\tau) + \frac{\eta_1-\Delta_1}{\xi^2\varepsilon^{2\alpha}v^{\varepsilon}}\right), \end{aligned}$$

where $\Delta_1 = \sqrt{\eta_1^2 - \xi^2\varepsilon^{2\alpha}v^{\varepsilon}ix(ix+1)}$.

By method of variable separation, we have

$$\frac{2dh_1(\tau)}{\left(h_1(\tau) + \frac{\eta_1+\Delta_1}{\xi^2\varepsilon^{2\alpha}v^{\varepsilon}}\right)\left(h_1(\tau) + \frac{\eta_1-\Delta_1}{\xi^2\varepsilon^{2\alpha}v^{\varepsilon}}\right)} = \xi^2\varepsilon^{2\alpha}v^{\varepsilon}d\tau.$$

Using partial fractions, we get

$$\frac{1}{\Delta_1}\left(\frac{1}{h_1(\tau) + \frac{\eta_1-\Delta_1}{\xi^2\varepsilon^{2\alpha}v^{\varepsilon}}} - \frac{1}{h_1(\tau) + \frac{\eta_1+\Delta_1}{\xi^2\varepsilon^{2\alpha}v^{\varepsilon}}}\right)dh_1(\tau) = d\tau.$$

Integrating both sides, we obtain

$$\log\left(\frac{h_1(\tau) + \frac{\eta_1-\Delta_1}{\xi^2\varepsilon^{2\alpha}v^{\varepsilon}}}{h_1(\tau) + \frac{\eta_1+\Delta_1}{\xi^2\varepsilon^{2\alpha}v^{\varepsilon}}}\right) = \Delta_1\tau + C.$$

Using boundary condition $h_1(\tau = 0) = 0$ we get $C = \log\left(\frac{\eta_1-\Delta_1}{\eta_1+\Delta_1}\right)$.

Solving for h_1, we obtain

$$h_1(\tau) = \frac{\left(\eta_1^2 - \Delta_1^2\right)\left(e^{\Delta_1\tau} - 1\right)}{\xi^2\varepsilon^{2\alpha}v^{\varepsilon}\left(\eta_1 + \Delta_1 - (\eta_1 - \Delta_1)e^{\Delta_1\tau}\right)}.$$

In order to solve $g_1(\tau)$ explicitly, we substitute h_1 into equation (5.4) and integrate with respect to τ on both sides. Then we get

$$g_1(\tau) \;=\; \left[(r - \lambda E_{\mathcal{M}}(Y_t))\,ix - \lambda E(Y_t)\right]\tau + \tau\lambda\int_{\Re}\left(e^{(ix+1)y} - 1\right)\phi_Y(y)dy$$

$$-\frac{2\omega}{\xi^2\varepsilon^{2\alpha}v^\varepsilon}\left[\log\left(1-\frac{(\Delta_1+\eta_1)+\left(1-e^{\Delta_1\tau}\right)}{2\Delta_1}\right)+(\Delta_1+\eta_1)\,\tau\right].$$

Proof of (ii). The details of the proof are similar to case (i). Hence, we have

$$f_2\left(\ell^\varepsilon,v^\varepsilon,t;y,t+\tau\right)=\exp\left(g_2\left(\tau\right)+v^\varepsilon h_2\left(\tau\right)+iy\ell^\varepsilon+r\tau\right),$$

where $g_2(\tau)$, $h_2(\tau)$, η_2 and Δ_2 are as given in the Lemma.

We can thus evaluate the characteristic functions in closed form. However, we are interested in the risk-neutral probabilities P_j. These can be inverted from the characteristic functions by performing the following integration

$$\begin{aligned}P_j\left(S_t,v_t^\varepsilon,t;K,T\right) &= P_j\left(\ell^\varepsilon,v^\varepsilon,t;\kappa,T\right)\\ &= \frac{1}{2}+\frac{1}{\pi}\int\limits_{0^+}^{+\infty}\textit{Re}\left[\frac{e^{-ix\kappa}f_j\left(\ell^\varepsilon,v^\varepsilon,t;x,T\right)}{ix}\right]dx\end{aligned}$$

for $j=1,2$, where $\ell^\varepsilon=\log S_t$, $v^\varepsilon=\log(v_t^\varepsilon)$, and $\kappa=\log(K)$.

To verify the above equation, firstly we note that

$$\begin{aligned}&E_{\mathcal{M}}\left[e^{ix(\log(S_t)-\log(K))}\mid\log(S_t)=L_t^\varepsilon,v_t^\varepsilon=v^\varepsilon\right]\\ =\;&E_{\mathcal{M}}\left[e^{ix(\ell^\varepsilon-\kappa)}\mid L_t^\varepsilon=\ell^\varepsilon,v_t^\varepsilon=v^\varepsilon\right]\\ =\;&\int\limits_{-\infty}^{+\infty}e^{ix(\ell^\varepsilon-\kappa)}dP_j\left(\ell^\varepsilon,v^\varepsilon,t;\kappa,T\right)\\ =\;&e^{-ix\kappa}\int\limits_{-\infty}^{+\infty}e^{ix\ell^\varepsilon}dP_j\left(\ell^\varepsilon,v^\varepsilon,t;\kappa,T\right)\\ =\;&e^{-ix\kappa}\int\limits_{-\infty}^{+\infty}e^{ix\kappa}(-\delta(\ell^\varepsilon-\kappa)d\kappa)\\ =\;&e^{-ix\kappa}f_j\left(\ell^\varepsilon,v_t^\varepsilon,t;x,T\right).\end{aligned}$$

Then

$$\begin{aligned}&\frac{1}{2}+\frac{1}{\pi}\int\limits_{0^+}^{+\infty}\textit{Re}\left[\frac{e^{-ix\kappa}f_j\left(\ell^\varepsilon,v_t^\varepsilon,t;x,T\right)}{ix}\right]dx\\ =\;&\frac{1}{2}+\frac{1}{\pi}\int\limits_{0^+}^{+\infty}\textit{Re}\left[\frac{E_{\mathcal{M}}\left[e^{ix(\log(S_t^\varepsilon)-\log(K))}\mid\log(S_t)=L_t^\varepsilon,v_t^\varepsilon=v^\varepsilon\right]}{ix}\right]dx\\ =\;&E_{\mathcal{M}}\left[\frac{1}{2}+\frac{1}{\pi}\int\limits_{0^+}^{+\infty}\textit{Re}\left[\frac{e^{ix(\ell^\varepsilon-\kappa)}}{ix}\right]dx\mid L_t^\varepsilon=\ell^\varepsilon,v_t^\varepsilon=v^\varepsilon\right]\\ =\;&E_{\mathcal{M}}\left[\frac{1}{2}+\frac{1}{\pi}\int\limits_{0^+}^{+\infty}\frac{\sin(x\left(\ell^\varepsilon-\kappa\right))}{x}dx\mid L_t^\varepsilon=\ell^\varepsilon,v_t^\varepsilon=v^\varepsilon\right]\\ =\;&E_{\mathcal{M}}\left[\frac{1}{2}+sgn\left(\ell^\varepsilon-\kappa\right)\frac{1}{\pi}\int\limits_{0^+}^{+\infty}\frac{\sin(x)}{x}dx\mid L_t^\varepsilon=\ell^\varepsilon,v_t^\varepsilon=v^\varepsilon\right]\end{aligned}$$

$$= E_{\mathcal{M}}\left[\frac{1}{2}+\frac{1}{2}sgn\left(\ell^{\varepsilon}-\kappa\right) \mid L_t^{\varepsilon}=\ell^{\varepsilon}, v_t^{\varepsilon}=v^{\varepsilon}\right]$$
$$= E_{\mathcal{M}}[1_{\ell^{\varepsilon}\geq\kappa} \mid L_t^{\varepsilon}=\ell^{\varepsilon}, v_t^{\varepsilon}=v^{\varepsilon}],$$

where we have used the Dirichlet formula $\int_{-\infty}^{+\infty}\frac{\sin(x)}{x}dx=1$, and the *sgn* function is defined as $sgn(x)=1$ if $x>0$, 0 if $x=0$ and -1 if $x<0$. □

In summary, we have just proved the following main theorem.

Theorem 5.1. *For each $\varepsilon>0$, the value of a European call option of SDE (2.5) is*

$$\widehat{C}\left(S_t^{\varepsilon}, v_t^{\varepsilon}, t; K, T\right)=S_t^{\varepsilon}P_1\left(S_t^{\varepsilon}, v_t^{\varepsilon}, t; K, T\right)-Ke^{-r(T-t)}P_2\left(S_t^{\varepsilon}, v_t^{\varepsilon}, t; K, T\right),$$

where P_1 and P_2 are as given in Lemma 2, and

$$\widehat{C}\left(S_t^{\varepsilon}, v_t^{\varepsilon}, t; K, T\right)=C\left(\log(S_t^{\varepsilon}), v^{\varepsilon}, t; \log(K), T\right).$$

Remark 5.2. In numerical computation, we firstly choose a real number $\varepsilon>0$ and then compute the value of $\widehat{C}\left(S_t^{\varepsilon}, v_t^{\varepsilon}, t; K, T\right)$ according to the formula as given in Theorem 3. The solution that we get is the value of a call option of the approximation model (2.5) and this value can be used as an approximating value of a call option of the fractional model (2.2) as ε approaches zero.

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