
**ON SOME I-CONVERGENT SEQUENCE SPACES DEFINED BY A SEQUENCE
OF MODULI**

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ABSTRACT. In this article we introduce the sequence spaces $c_0^I(F)$, $c^I(F)$ and $l_\infty^I(F)$ for the sequence of moduli $F = (f_k)$ and study some of the properties of these spaces. The results here in proved are analogous to those by Vakeel.A.Khan and Khalid Ebadullah [Theory and Applications of Mathematics and Computer Science,1(2)(2011): 22-30].

KEYWORDS : Ideal; Filter; Sequence of moduli; Lipschitz function; I-convergence field; I-convergent; Monotone; Solid spaces

1. INTRODUCTION

Throughout the article \mathbb{N} , \mathbb{R} , \mathbb{C} and ω denotes the set of natural, real, complex numbers and the class of all sequences respectively.

The notion of the statistical convergence was introduced by H.Fast [5]. Later on it was studied by J.A.Fridy [6, 7] from the sequence space point of view and linked it with the summability theory.

The notion of I-convergence is a generalization of the statistical convergence. At the initial stage it was studied by Kostyrko, Šalát and Wilezyński [18]. Later on it was studied by Šalát [26], Tripathy and Ziman [32] and Demirci [3], Das, Kostyrko, Wilczynski, and Malik [2], Mursaleen and Alotaibi [23], Mursaleen, Mohiuddine, and Edely [24], Mursaleen, and Mohiuddine [25], Mursaleen and Mohiuddine [26], Sahiner, Gurdal, Saltan and Gunawan [33] and Kumar [19]. Here we give some preliminaries about the notion of I-convergence.

Let X be a non empty set. Then a family of sets $I \subseteq 2^X$ (power set of X) is said to be an ideal if I is additive i.e., $A, B \in I \Rightarrow A \cup B \in I$ and hereditary i.e., $A \in I, B \subseteq A \Rightarrow B \in I$.

A non-empty family of sets $\mathcal{L}(I) \subseteq 2^X$ is said to be filter on X if and only if $\Phi \notin \mathcal{L}(I)$, for $A, B \in \mathcal{L}(I)$ we have $A \cap B \in \mathcal{L}(I)$ and for each $A \in \mathcal{L}(I)$ and $A \subseteq B$ implies $B \in \mathcal{L}(I)$.

An Ideal $I \subseteq 2^X$ is called non-trivial if $I \neq 2^X$.

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A non-trivial ideal $I \subseteq 2^X$ is called admissible if $\{\{x\} : x \in X\} \subseteq I$.

A non-trivial ideal I is maximal if there cannot exist any non-trivial ideal $J \neq I$ containing I as a subset.

For each ideal I , there is a filter $\mathcal{L}(I)$ corresponding to I .

i.e., $\mathcal{L}(I) = \{K \subseteq N : K^c \in I\}$, where $K^c = N - K$.

The idea of modulus was structured in 1953 by Nakano. (See [27]). A function $f : [0, \infty) \rightarrow [0, \infty)$ is called a modulus if

- (1) $f(t) = 0$ if and only if $t = 0$,
- (2) $f(t+u) \leq f(t) + f(u)$ for all $t, u \geq 0$,
- (3) f is increasing, and
- (4) f is continuous from the right at zero.

Ruckle [28, 29, 30] used the idea of a modulus function f to construct the sequence space

$$X(f) = \{x = (x_k) : (f(|x_k|)) \in X\}.$$

This space is an FK space, and Ruckle [28] proved that the intersection of all such $X(f)$ spaces is ϕ , the space of all finite sequences.

The space $X(f)$ is closely related to the space l_1 which is an $X(f)$ space with $f(x) = x$ for all real $x \geq 0$. Thus Ruckle [29] proved that, for any modulus f ,

$$X(f) \subset l_1 \text{ and } X(f)^\alpha = l_\infty$$

Where

$$X(f)^\alpha = \{y = (y_k) \in \omega : \sum_{k=1}^{\infty} f(|y_k x_k|) < \infty\}.$$

The space $X(f)$ is a Banach space with respect to the norm

$$\|x\| = \sum_{k=1}^{\infty} f(|x_k|) < \infty. \text{ (See [29]).}$$

Spaces of the type $X(f)$ are a special case of the spaces structured by B. Gramsch [10]. From the point of view of local convexity, spaces of the type $X(f)$ are quite pathological. Therefore symmetric sequence spaces, which are locally convex have been frequently studied by D.J.H. Garling [8, 9], G. Köthe [17] and W.H. Ruckle [28, 29, 30].

After then E. Kolk gave an extension of $X(f)$ by considering a sequence of moduli $F = (f_k)$ and defined the sequence space

$$X(F) = \{x = (x_k) : (f_k(|x_k|)) \in X\}. \text{ (See [15, 16]).}$$

Definition 1.1. A sequence space E is said to be solid or normal if $(x_k) \in E$ implies $(\alpha_k x_k) \in E$ for all sequence of scalars (α_k) with $|\alpha_k| < 1$ for all $k \in \mathbb{N}$.

Definition 1.2. A sequence space E is said to be monotone if it contains the canonical preimages of all its stepspaces.

Definition 1.3. A sequence space E is said to be convergence free if $(y_k) \in E$ whenever $(x_k) \in E$ and $x_k = 0$ implies $y_k = 0$.

Definition 1.4. A sequence space E is said to be a sequence algebra if $(x_k y_k) \in E$ whenever $(x_k) \in E$, $(y_k) \in E$.

Definition 1.5. A sequence space E is said to be symmetric if $(x_{\pi(k)}) \in E$ whenever $(x_k) \in E$ where $\pi(k)$ is a permutation on \mathbb{N} .

Definition 1.6. A sequence $(x_k) \in \omega$ is said to be I-convergent to a number L if for every $\epsilon > 0$, $\{k \in \mathbb{N} : |x_k - L| \geq \epsilon\} \in I$. In this case we write $I - \lim x_k = L$. The space c^I of all I-convergent sequences to L is given by

$$c^I = \{(x_k) \in \omega : \{k \in \mathbb{N} : |x_k - L| \geq \epsilon\} \in I, \text{ for some } L \in \mathbb{Q}\}.$$

Definition 1.7. A sequence $(x_k) \in \omega$ is said to be I-null if $L = 0$. In this case we write $I - \lim x_k = 0$.

Definition 1.8. A sequence $(x_k) \in \omega$ is said to be I-cauchy if for every $\epsilon > 0$ there exists a number $m = m(\epsilon)$ such that $\{k \in \mathbb{N} : |x_k - x_m| \geq \epsilon\} \in I$.

Definition 1.9. A sequence $(x_k) \in \omega$ is said to be I-bounded if there exists $M > 0$ such that $\{k \in \mathbb{N} : |x_k| > M\} \in I$.

Definition 1.10. Take for I the class I_f of all finite subsets of \mathbb{N} . Then I_f is a non-trivial admissible ideal and I_f convergence coincides with the usual convergence with respect to the metric in X. (see [18]).

Definition 1.11. For $I = I_\delta$ the class of all $A \subset \mathbb{N}$ with $\delta(A) = 0$ respectively. I_δ is a non-trivial admissible ideal, I_δ -convergence is said to be logarithmic statistical convergence. (see [18]).

Definition 1.12. A map h defined on a domain $D \subset X$ i.e., $h : D \subset X \rightarrow R$ is said to satisfy Lipschitz condition if $|h(x) - h(y)| \leq K|x - y|$ where K is known as the Lipschitz constant. The class of K-Lipschitz functions defined on D is denoted by $h \in (D, K)$ (see [32]).

Definition 1.13. A convergence field of I-convergence is a set

$$F(I) = \{x = (x_k) \in l_\infty : \text{there exists } I - \lim x \in R\}.$$

The convergence field $F(I)$ is a closed linear subspace of l_∞ with respect to the supremum norm, $F(I) = l_\infty \cap c^I$ (See [31]).

Define a function $h : F(I) \rightarrow R$ such that $h(x) = I - \lim x$, for all $x \in F(I)$, then the function $h : F(I) \rightarrow R$ is a Lipschitz function. (see [1, 4, 11, 12, 13, 14, 20, 21, 22, 26, 31, 34, 35]).

Throughout the article $l_\infty, c^I, c_0^I, m^I$ and m_0^I represent the bounded, I-convergent, I-null, bounded I-convergent and bounded I-null sequence spaces respectively.

In this article we introduce the following classes of sequence spaces.

$$\begin{aligned} c^I(F) &= \{(x_k) \in \omega : I - \lim f_k(|x_k|) = L \text{ for some } L\} \in I \\ c_0^I(F) &= \{(x_k) \in \omega : I - \lim f_k(|x_k|) = 0\} \in I \\ l_\infty^I(F) &= \{(x_k) \in \omega : \sup_k f_k(|x_k|) < \infty\} \in I \end{aligned}$$

We also denote by

$$m^I(F) = c^I(F) \cap l_\infty(F)$$

and

$$m_0^I(F) = c_0^I(F) \cap l_\infty(F)$$

The following Lemmas will be used for establishing some results of this article.

Lemma 1.14. Let E be a sequence space. If E is solid then E is monotone.

Lemma 1.15. Let $K \in \mathcal{L}(I)$ and $M \subseteq N$. If $M \notin I$, then $M \cap N \notin I$.

Lemma 1.16. If $I \subset 2^{\mathbb{N}}$ and $M \subseteq N$. If $M \notin I$, then $M \cap N \notin I$.

2. MAIN RESULTS

Theorem 2.1. For any sequence of moduli $F = (f_k)$, the classes of sequences $c^I(F)$, $c_0^I(F)$, $m^I(F)$ and $m_0^I(F)$ are linear spaces.

Proof: We shall prove the result for the space $c^I(F)$.

The proof for the other spaces will follow similarly.

Let $(x_k), (y_k) \in c^I(F)$ and let α, β be the scalars. Then

$$I - \lim f_k(|x_k - L_1|) = 0, \text{ for some } L_1 \in c;$$

$$I - \lim f_k(|y_k - L_2|) = 0, \text{ for some } L_2 \in c.$$

That is for a given $\epsilon > 0$, we have

$$A_1 = \{k \in \mathbb{N} : f_k(|x_k - L_1|) > \frac{\epsilon}{2}\} \in I, \quad (1)$$

$$A_2 = \{k \in \mathbb{N} : f_k(|y_k - L_2|) > \frac{\epsilon}{2}\} \in I. \quad (2)$$

Since f_k is a modulus function, we have

$$\begin{aligned} f_k(|(\alpha x_k + \beta y_k) - (\alpha L_1 + \beta L_2)|) &\leq f_k(|\alpha||x_k - L_1|) + f_k(|\beta||y_k - L_2|) \\ &\leq f_k(|x_k - L_1|) + f_k(|y_k - L_2|). \end{aligned}$$

Now, by (1) and (2), $\{k \in \mathbb{N} : f_k(|(\alpha x_k + \beta y_k) - (\alpha L_1 + \beta L_2)|) > \epsilon\} \subset A_1 \cup A_2$.

Therefore $(\alpha x_k + \beta y_k) \in c^I(F)$.

Hence $c^I(F)$ is a linear space.

Theorem 2.2. A sequence $x = (x_k) \in m^I(F)$ I -converges if and only if for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that

$$\{k \in \mathbb{N} : f_k(|x_k - x_{N_\epsilon}|) < \epsilon\} \in m^I(F). \quad (3)$$

Proof: Suppose that $L = I - \lim x$. Then

$$B_\epsilon = \{k \in \mathbb{N} : |x_k - L| < \frac{\epsilon}{2}\} \in m^I(F). \text{ For all } \epsilon > 0.$$

Fix an $N_\epsilon \in B_\epsilon$. Then we have

$$|x_{N_\epsilon} - x_k| \leq |x_{N_\epsilon} - L| + |L - x_k| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

which holds for all $k \in B_\epsilon$.

Hence $\{k \in \mathbb{N} : f_k(|x_k - x_{N_\epsilon}|) < \epsilon\} \in m^I(F)$.

Conversely, suppose that $\{k \in \mathbb{N} : f_k(|x_k - x_{N_\epsilon}|) < \epsilon\} \in m^I(F)$.

That is $\{k \in \mathbb{N} : (|x_k - x_{N_\epsilon}|) < \epsilon\} \in m^I(F)$ for all $\epsilon > 0$. Then the set

$$C_\epsilon = \{k \in \mathbb{N} : x_k \in [x_{N_\epsilon} - \epsilon, x_{N_\epsilon} + \epsilon]\} \in m^I(F) \text{ for all } \epsilon > 0.$$

Let $J_\epsilon = [x_{N_\epsilon} - \epsilon, x_{N_\epsilon} + \epsilon]$. If we fix an $\epsilon > 0$ then we have $C_\epsilon \in m^I(F)$ as well as $C_{\frac{\epsilon}{2}} \in m^I(F)$. Hence $C_\epsilon \cap C_{\frac{\epsilon}{2}} \in m^I(F)$. This implies that

$$J = J_\epsilon \cap J_{\frac{\epsilon}{2}} \neq \phi,$$

that is

$$\{k \in \mathbb{N} : x_k \in J\} \in m^I(F),$$

that is

$$\text{diam}J \leq \text{diam}J_\epsilon,$$

where the diam of J denotes the length of interval J.

In this way, by induction we get the sequence of closed intervals

$$J_\epsilon = I_0 \supseteq I_1 \supseteq \dots \supseteq I_k \supseteq \dots$$

with the property that $\text{diam}I_k \leq \frac{1}{2} \text{diam}I_{k-1}$ for $(k=2,3,4,\dots)$ and $\{k \in \mathbb{N} : x_k \in I_k\} \in m^I(F)$ for $(k=1,2,3,4,\dots)$.

Then there exists a $\xi \in \cap I_k$ where $k \in \mathbb{N}$ such that $\xi = I - \lim x$. So that $f_k(\xi) = I - \lim f_k(x)$, that is $L = I - \lim f_k(x)$.

Result 2.3. The spaces $c_0^I(F)$ and $m_0^I(F)$ are solid and monotone .

Proof:We shall prove the result for $c_0^I(F)$. Let $x_k \in c_0^I(F)$.Then

$$I - \lim_k f_k(|x_k|) = 0. \tag{4}$$

Let (α_k) be a sequence of scalars with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$. Then the result follows from (4) and the following inequality

$$f_k(|\alpha_k x_k|) \leq |\alpha_k| f_k(|x_k|) \leq f_k(|x_k|) \text{ for all } k \in \mathbb{N}.$$

That the space $c_0^I(F)$ is monotone follows from the Lemma 1.14.

For $m_0^I(F)$ the result can be proved similarly.

Result 2.4.The spaces $c^I(F)$ and $m^I(F)$ are neither solid nor monotone in general .

Proof:Here we give a counter example.

Let $I = I_\delta$ and $f(x) = x^2$ for all $x \in [0, \infty)$. Consider the K-step space $X_K(f)$ of X defined as follows,

Let $(x_k) \in X$ and let $(y_k) \in X_K$ be such that

$$(y_k) = \begin{cases} (x_k), & \text{if } k \text{ is even,} \\ 0, & \text{otherwise.} \end{cases}$$

Consider the sequence (x_k) defined by $(x_k) = 1$ for all $k \in \mathbb{N}$.

Then $(x_k) \in c^I(F)$ but its K-stepspace preimage does not belong to $c^I(F)$. Thus $c^I(F)$ is not monotone.Hence $c^I(F)$ is not solid.

Result 2.5. The spaces $c^I(F)$ and $c_0^I(F)$ are sequence algebras.

Proof: We prove that $c_0^I(F)$ is a sequence algebra.

Let $(x_k), (y_k) \in c_0^I(F)$. Then

$$I - \lim f_k(|x_k|) = 0$$

and

$$I - \lim f_k(|y_k|) = 0.$$

Then we have

$$I - \lim f_k(|(x_k \cdot y_k)|) = 0.$$

Thus $(x_k, y_k) \in c_0^I(F)$ is a sequence algebra.
For the space $c^I(F)$, the result can be proved similarly.

Result 2.6. The spaces $c^I(F)$ and $c_0^I(F)$ are not convergence free in general.

Proof: Here we give a counter example.

Let $I = I_f$ and $f(x) = x^3$ for all $x \in [0, \infty)$. Consider the sequence (x_k) and (y_k) defined by

$$x_k = \frac{1}{k} \quad \text{and } y_k = k \text{ for all } k \in \mathbb{N}.$$

Then $(x_k) \in c^I(F)$ and $c_0^I(F)$, but $(y_k) \notin c^I(F)$ and $c_0^I(F)$.

Hence the spaces $c^I(F)$ and $c_0^I(F)$ are not convergence free.

Result 2.7. If I is not maximal and $I \neq I_f$, then the spaces $c^I(F)$ and $c_0^I(F)$ are not symmetric.

Proof: Let $A \in I$ be infinite and $f(x) = x$ for all $x \in [0, \infty)$.

If

$$x_k = \begin{cases} 1, & \text{for } k \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Then by lemma 1.16 $x_k \in c_0^I(F) \subset c^I(F)$.

Let $K \subset \mathbb{N}$ be such that $K \notin I$ and $\mathbb{N} - K \notin I$. Let $\phi : K \rightarrow A$ and $\psi : \mathbb{N} - K \rightarrow \mathbb{N} - A$ be bijections, then the map $\pi : \mathbb{N} \rightarrow \mathbb{N}$ defined by

$$\pi(k) = \begin{cases} \phi(k), & \text{for } k \in K, \\ \psi(k), & \text{otherwise,} \end{cases}$$

is a permutation on \mathbb{N} , but $x_{\pi(k)} \notin c^I(F)$ and $x_{\pi(k)} \notin c_0^I(F)$.

Hence $c_0^I(F)$ and $c^I(F)$ are not symmetric.

Theorem 2.3. Let $F = (f_k)$ be the sequence of moduli. Then $c_0^I(F) \subset c^I(F) \subset l_\infty^I(F)$ and the inclusions are proper.

Proof: Let $x_k \in c^I(F)$. Then there exists $L \in \mathbb{C}$ such that

$$I - \lim f_k(|x_k - L|) = 0.$$

We have $f_k(|x_k|) \leq \frac{1}{2}f_k(|x_k - L|) + f_k(\frac{1}{2}|L|)$.

Taking the supremum over k on both sides we get $x_k \in l_\infty(F)$.

The inclusion $c_0^I(F) \subset c^I(F)$ is obvious.

Theorem 2.4. The function $h : m^I(F) \rightarrow \mathbb{R}$ is the Lipschitz function, where $m^I(F) = c^I(F) \cap l_\infty(F)$, and hence uniformly continuous.

Proof: Let $x, y \in m^I(F)$, $x \neq y$. Then the sets

$$A_x = \{k \in \mathbb{N} : |x_k - h(x)| \geq \|x - y\|\} \in I,$$

$$A_y = \{k \in \mathbb{N} : |y_k - h(y)| \geq \|x - y\|\} \in I.$$

Thus the sets,

$$B_x = \{k \in \mathbb{N} : |x_k - h(x)| < \|x - y\|\} \in m^I(F),$$

$$B_y = \{k \in \mathbb{N} : |y_k - h(y)| < \|x - y\|\} \in m^I(F).$$

Hence also $B = B_x \cap B_y \in m^I(F)$, so that $B \neq \phi$.

Now taking k in B ,

$$|\hbar(x) - \hbar(y)| \leq |\hbar(x) - x_k| + |x_k - y_k| + |y_k - \hbar(y)| \leq 3\|x - y\|.$$

Thus \hbar is a Lipschitz function. For $m_0^I(F)$ the result can be proved similarly.

Result 2.10. If $x, y \in m^I(F)$, then $(x.y) \in m^I(F)$ and $\hbar(xy) = \hbar(x)\hbar(y)$.

Proof: For $\epsilon > 0$

$$B_x = \{k \in \mathbb{N} : |x_k - \hbar(x)| < \epsilon\} \in m^I(F),$$

$$B_y = \{k \in \mathbb{N} : |y_k - \hbar(y)| < \epsilon\} \in m^I(F).$$

Now,

$$\begin{aligned} |x_k y_k - \hbar(x)\hbar(y)| &= |x_k y_k - x_k \hbar(y) + x_k \hbar(y) - \hbar(x)\hbar(y)| \\ &\leq |x_k| |y_k - \hbar(y)| + |\hbar(y)| |x_k - \hbar(x)|. \end{aligned} \quad (8)$$

As $m^I(F) \subseteq l_\infty(F)$, there exists an $M \in \mathbb{R}$ such that $|x_k| < M$ and $|\hbar(y)| < M$.

Using (8) we get

$$|x_k y_k - \hbar(x)\hbar(y)| \leq M\epsilon + M\epsilon = 2M\epsilon.$$

For all $k \in B_x \cap B_y \in m^I(F)$. Hence $(x.y) \in m^I(F)$ and $\hbar(xy) = \hbar(x)\hbar(y)$.

For $m_0^I(F)$ the result can be proved similarly.

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