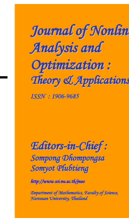

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A NEW ITERATIVE METHOD FOR NONEXPANSIVE MAPPINGS IN HILBERT SPACES

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ABSTRACT. In this paper, a new iterative method for finding a common fixed point of a countable nonexpansive mappings in a Hilbert space, is introduced. Then a strong convergence theorem for a countable family of nonexpansive mappings is proved. This theorem improve and extend some recent results of Tian (2010) and Xu (2004).

KEYWORDS : Fixed point; Nonexpansive mapping; Iterative method; Variational inequality; Viscosity approximation.

1. INTRODUCTION

Let H be a real Hilbert space. A mapping S of H into itself is called nonexpansive, if $\|Sx - Sy\| \leq \|x - y\|$ for all $x, y \in H$. Let $Fix(S)$ denote the fixed points set of S. We assume $Fix(S) \neq \emptyset$, it is well known, $Fix(S)$ is closed and convex. Recall that a contraction on H is a self-mapping f of H such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|$ for all $x, y \in H$, where $\alpha \in (0, 1)$ is a constant. Let A be a bounded linear operator on H. A is strongly positive; that is, there exists a constant $\overline{\gamma} > 0$ such that $\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2$, for all $x \in H$.

Moudafi [5] introduced the viscosity approximation method for nonexpansive mappings. Let f be a contraction on H, starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Sx_n, \; n \geq 0, \tag{1.1}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$. Xu [9] proved that under certain appropriate conditions on $\{\alpha_n\}$, the sequence $\{x_n\}$ generated by (1.1) converges strongly to the unique solution x^* in $Fix(S)$ of the variational inequality:

$$\langle (I - f)x^*, x^* - x\rangle \leq 0, \; \text{for all } x \in Fix(S). \tag{1.2}$$

Note that iterative methods for nonexpansive mappings can be used to solve a convex minimization problem. See, e.g., [2, 8, 10] and references therein. A typical

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problem is that of minimizing a quadratic function over the set of the fixed points of nonexpansive mapping on a real Hilbert space

$$\min_{x\in Fix(S)} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle, \tag{1.3}$$

where b is a given point in H.

In [8], it is proved, the sequence $\{x_n\}$ defined by the iterative method below with an arbitrary initial $x_0 \in H$

$$x_{n+1} = \alpha_n b + (I - \alpha_n A)Sx_n,\ n \geq 0, \tag{1.4}$$

converges strongly to the unique solution of the minimization problem (1.3) provided the sequence $\{\alpha_n\}$ satisfies certain conditions. Combining the iterative method (1.1) and (1.4), Marino and Xu [4] consider the following general iterative method:

$$x_{n+1} = \alpha_n\gamma f(x_n) + (I - \alpha_n A)Sx_n,\ n \geq 0. \tag{1.5}$$

It is proved, if the sequence $\{\alpha_n\}$ satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.5) converges strongly to the unique solution of the variational inequality

$$\langle(\gamma f - A)x^*, x - x^*\rangle \leq 0,\ \text{for all } x \in Fix(S).$$

which is the optimality condition for the minimization problem

$$\min_{x\in Fix(S)} \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

On the other hand, Yamada [10] introduced the following hybrid iterative method for solving the variational inequality

$$x_{n+1} = Sx_n - \mu\lambda_n F(Sx_n),\ n \geq 0, \tag{1.6}$$

where F is $k - Lipschitzian$ and η-strongly monotone operator with $k > 0, \eta > 0$ and $0 < \mu < 2\eta/k^2$, then, if $\{\lambda_n\}$ satisfies appropriate conditions, the sequence $\{x_n\}$ generated by (1.6) converges strongly to the unique solution of the variational inequality

$$\langle Fx^*, x - x^*\rangle \geq 0,\ \text{for all } x \in Fix(S).$$

Tian [7] combined the iterative method (1.5) with the Yamada's method (1.6) and considered the following iterative method:

$$x_{n+1} = \alpha_n\gamma f(x_n) + (I - \mu\alpha_n F)Sx_n,\ n \geq 0. \tag{1.7}$$

He proved, if the sequence $\{\alpha_n\}$ satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.7) converges strongly to the unique solution $x^* \in Fix(S)$ of the variational inequality

$$\langle(\gamma f - \mu F)x^*, x - x^*\rangle \leq 0,\ \text{for all } x \in Fix(S).$$

In this paper, motivated by Tian [7], we prove a strong convergence theorem for a countable family of nonexpansive mappings in a Hilbert space. Our result improve and extend the corresponding results in recent works.

2. PRELIMINARIES

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and the norm $\|.\|$. Weak and strong convergence is denoted by notation $\rightharpoonup$ and $\rightarrow$, respectively. In a real Hilbert space H,

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2,$$

for all $x, y \in H$ and $\lambda \in \mathbb{R}$. Let C be a nonempty closed convex subset of H. Then, for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| \leq \|x - y\|, \quad \text{for all } y \in C.$$

Such a P_C is called the metric projection of H onto C. It is known, P_C is nonexpansive. Further, for $x \in H$ and $z \in C$,

$$z = P_C(x) \iff \langle x - z, z - y \rangle \geq 0, \quad \text{for all } y \in C.$$

Now, we collect some lemmas which will be used in the main result.

Lemma 2.1. *Let H be a real Hilbert space. Then, for all $x, y \in H$,*

(I) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y \rangle$;
(II) $\|x+y\|^2 \geq \|x\|^2 + 2\langle y, x \rangle$.

Lemma 2.2. [1] *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n v_n + r_n,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$, $\{r_n\}$ is a sequence of nonnegative real numbers and $\{v_n\}$ is a sequence in $\mathbb{R}$ such that

(I) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(II) $\limsup_{n\to\infty} v_n \leq 0$;
(III) $\sum_{n=1}^{\infty} r_n < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.3. [3] *Let C be a nonempty closed convex subset of H and $S : C \to C$ a nonexpansive mapping with $Fix(S) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I-S)(x_n)\}$ converges strongly to y, then $(I-S)x = y$.*

Lemma 2.4. [1] *Let C be a nonempty closed convex subset of H. Suppose*

$$\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in C\} < \infty.$$

Then, for each $y \in C$, $\{T_n y\}$ converges strongly to some point of C. Moreover, let T be a mapping of C into itself defined by $Ty = \lim_{n\to\infty} T_n y$, for all $y \in C$. Then $\lim_{n\to\infty} \sup\{\|Tz - T_n z\| : z \in C\} = 0$.

Theorem 2.5. [7] *Let H be a real Hilbert space, $S : H \to H$ be a nonexpansive mapping with $Fix(S) \neq \emptyset$, $f : H \to H$ be a contraction with coefficient $0 < \alpha < 1$ and F be a k-Lipschitzian and η-strongly monotone operator on H with $k > 0, \eta > 0$. Let $0 < \mu < 2\eta/k^2$ and $0 < \gamma < \mu(\eta - \frac{\mu k^2}{2})/\alpha = \tau/\alpha$. Then the unique fixed point $x_t \in H$ of the contraction $x \mapsto t\gamma f(x) + (I - t\mu F)Sx$ converges strongly to a fixed point q of S as $t \to 0$ which solves the following variational inequality*

$$\langle (\mu F - \gamma f)q, q - z) \rangle \leq 0, \text{ for all } z \in Fix(S).$$

Theorem 2.6. [7] *Let H be a real Hilbert space, $S : H \to H$ be a nonexpansive mapping with $Fix(S) \neq \emptyset$, $f : H \to H$ be a contraction with coefficient $0 < \alpha < 1$ and F be a k-Lipschitzian and η-strongly monotone operator on H with $k > 0, \eta > 0$. Let $0 < \mu < 2\eta/k^2$ and $0 < \gamma < \mu(\eta - \frac{\mu k^2}{2})/\alpha = \tau/\alpha$. Suppose $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying the following conditions:*

(I) $\lim_{n\to\infty} \alpha_n = 0$;
(II) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(III) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$.

Let $x_0 \in H$. Then the sequence $\{x_n\}$ defined by (1.7) converges strongly to q that is obtained in Theorem 2.5.

3. MAIN RESULTS

In this section, we prove the following strong convergence theorem for finding a common element of fixed points set of a countable family of nonexpansive mappings in a Hilbert space.

Theorem 3.1. *Let H be a real Hilbert space H. Let $\{S_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive self-mappings on H which satisfies $\bigcap_{n=1}^{\infty} Fix(S_n) \neq \emptyset$. Let f be a contraction of H into itself with coefficient $0 < \alpha < 1$ and F a k-Lipschitzian and η-strongly monotone operator on H with $k > 0, \eta > 0$. Let $0 < \mu < 2\eta/k^2$, $0 < \gamma < \mu(\eta - \frac{\mu k^2}{2})/\alpha = \tau/\alpha$ and $\tau < 1$. Define a sequence $\{x_n\} \subset H$ as follows: $x_1 = x \in H$ and*

$$\begin{cases} y_n = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu F) S_n x_n, \\ x_{n+1} = (1 - \beta_n) y_n + \beta_n S_n y_n, \quad n \geq 1, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0, 1]$ satisfying the following conditions:

(I) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(II) $\lim_{n\to\infty} \beta_n = 0$ *or* $\beta_n \in [0, b)$ *for some* $b \in (0, 1)$ *and* $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;
(III) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$.

Suppose $\sum_{n=1}^{\infty} \sup\{\|S_{n+1}z - S_n z\| : z \in K\} < \infty$ for any bounded subset K of H. Let S be a mapping of H into itself defined by $Sz = \lim_{n\to\infty} S_n z$ for all $z \in H$ and suppose $Fix(S) = \bigcap_{n=1}^{\infty} Fix(S_n)$. Then the sequences $\{x_n\}$ defined by (3.1) converge strongly to $q \in Fix(S)$ which is a unique solution of the following variational inequality

$$\langle (\mu F - \gamma f)q, q - z \rangle \leq 0, \text{ for all } z \in Fix(S).$$

Proof. Let $Q = P_{\bigcap_{n=1}^{\infty} Fix(S_n)}$. So

$$\begin{aligned} \|Q(I - \mu F + \gamma f)(x) - Q(I - \mu F + \gamma f)(y)\| \leq & \ \|(I - \mu F + \gamma f)(x) - \\ & \ (I - \mu F + \gamma f)(y)\| \\ \leq & \ \|(I - \mu F)(x) - (I - \mu F)(y)\| \\ & + \gamma \|f(x) - f(y)\| \\ \leq & \ (1 - \tau)\|x - y\| + \gamma \alpha \|x - y\| \\ = & \ (1 - (\tau - \gamma\alpha))\|x - y\|, \end{aligned}$$

for all $x, y \in H$. Therefore, $P_{\bigcap_{n=1}^{\infty} Fix(S_n)}(I - \mu F + \gamma f)$ is a contraction of H into itself, which implies, there exists a unique element $q \in H$ such that $q = Q(I - \mu F + \gamma f)(q) = P_{\bigcap_{n=1}^{\infty} Fix(S_n)}(I - \mu F + \gamma f)(q)$ or equivalently

$$\langle (\mu F - \gamma f)q, q - z) \rangle \leq 0, \text{ for all } z \in Fix(S). \tag{3.2}$$

We proceed with following steps:
Step 1. $\{x_n\}$ and $\{y_n\}$ are bounded. Let $p \in Fix(S)$. Then, from (3.1), we have

$$\begin{aligned} \|x_{n+1}-p\| = & \ \|(1-\beta_n)(y_n-p)+\beta_n(S_ny_n-p)\| \leq \|y_n-p\| \\ = & \ \|\alpha_n(\gamma f(x_n)-\mu F(p))+(I-\alpha_n\mu F)(S_nx_n-p)\| \\ \leq & \ (1-\alpha_n\tau)\|x_n-p\|+\alpha_n\gamma\alpha\|x_n-p\|+\alpha_n\|\gamma f(p)-\mu F(p)\| \\ \leq & \ (1-\alpha_n(\tau-\gamma\alpha))\|x_n-p\|+\alpha_n\|\gamma f(p)-\mu F(p)\| \\ \leq & \ \max\{\|x_n-p\|, \tfrac{\|\gamma f(p)-\mu F(p)\|}{\tau-\gamma\alpha}\}. \end{aligned}$$

By induction,

$$\|x_n-p\| \leq \max\{\|x_1-p\|, \frac{\|\gamma f(p)-\mu F(p)\|}{\tau-\gamma\alpha}\}, \quad n \geq 1.$$

Hence, $\{x_n\}$ is bounded, so are $\{y_n\}$, $\{f(x_n)\}$, $\{(FS_n)x_n\}$ and $\{S_ny_n\}$. Without loss of generality, we may assume $\{x_n\}$, $\{y_n\}$, $\{f(x_n)\}$, $\{(FS_n)x_n\}$, $\{S_ny_n\} \subset K$, where K is a bounded set of H.

Step 2. $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$. Since K is bounded, $\{S_ny_n-y_n\}$, $\{f(x_n)\}$ and $\{(FS_n)x_n\}$ are bounded. Let

$$M_1 = \sup\{\|S_ny_n-y_n\|, \|f(x_n)\|, \|(\mu FS_n)x_n\| : n \in \mathbb{N}\}.$$

From the definition of $\{x_n\}$, it is easily seen

$$\begin{aligned} \|x_{n+2}-x_{n+1}\| = & \ \|(1-\beta_{n+1})y_{n+1}+\beta_{n+1}S_{n+1}y_{n+1}-(1-\beta_n)y_n-\beta_nS_ny_n\| \\ = & \ \|(1-\beta_{n+1})y_{n+1}+\beta_{n+1}S_{n+1}y_{n+1}-(1-\beta_{n+1})y_n-\beta_nS_ny_n \\ & \ +(1-\beta_{n+1})y_n-(1-\beta_n)y_n-\beta_{n+1}S_ny_n+\beta_{n+1}S_ny_n\| \\ = & \ \|(1-\beta_{n+1})(y_{n+1}-y_n)+\beta_{n+1}(S_{n+1}y_{n+1}-S_ny_n) \\ & \ +(\beta_{n+1}-\beta_n)(S_ny_n-y_n)\| \\ \leq & \ (1-\beta_{n+1})\|y_{n+1}-y_n\|+\beta_{n+1}\|S_{n+1}y_{n+1}-S_ny_n\| \\ & \ +|\beta_{n+1}-\beta_n|M_1 \\ \leq & \ (1-\beta_{n+1})\|y_{n+1}-y_n\|+\beta_{n+1}\|S_{n+1}y_{n+1}-S_ny_{n+1}\| \\ & \ +\beta_{n+1}\|y_{n+1}-y_n\|+|\beta_{n+1}-\beta_n|M_1 \\ \leq & \ \|y_{n+1}-y_n\|+\|S_{n+1}y_{n+1}-S_ny_{n+1}\|+|\beta_{n+1}-\beta_n|M_1, \end{aligned} \tag{3.3}$$

for all $n \in \mathbb{N}$. From (3.1), we obtain

$$\begin{aligned} \|y_{n+1}-y_n\| = & \ \|\alpha_{n+1}\gamma f(x_{n+1})+(I-\alpha_{n+1}\mu F)S_{n+1}x_{n+1}-\alpha_n\gamma f(x_n) \\ & \ -(I-\alpha_n\mu F)S_nx_n\| \\ = & \ \|(I-\alpha_{n+1}\mu F)(S_{n+1}x_{n+1}-S_nx_n)-(\alpha_{n+1}-\alpha_n)\mu F(S_nx_n) \\ & \ +\alpha_{n+1}\gamma(f(x_{n+1})-f(x_n))+(\alpha_{n+1}-\alpha_n)\gamma f(x_n)\| \\ \leq & \ (1-\alpha_{n+1}\tau)\|S_{n+1}x_{n+1}-S_nx_n\|+\alpha_{n+1}\gamma\alpha\|x_{n+1}-x_n\| \\ & \ +|\alpha_{n+1}-\alpha_n|\|\gamma f(x_n)-\mu F(S_nx_n)\| \\ \leq & \ (1-\alpha_{n+1}\tau)(\|S_{n+1}x_{n+1}-S_{n+1}x_n\|+\|S_{n+1}x_n-S_nx_n\|) \\ & \ +\alpha_{n+1}\gamma\alpha\|x_{n+1}-x_n\|+|\alpha_{n+1}-\alpha_n|(\gamma+1)M_1 \\ \leq & \ (1-\alpha_{n+1}(\tau-\gamma\alpha))\|x_{n+1}-x_n\|+M_1(\gamma+1)|\alpha_{n+1}-\alpha_n| \\ & \ +\|S_{n+1}x_n-S_nx_n\|, \end{aligned} \tag{3.4}$$

for all $n \in N$. Using (3.4) in (3.3), we have

$$\begin{aligned} \|x_{n+2}-x_{n+1}\| \leq & \ (1-\alpha_{n+1}(\tau-\gamma\alpha))\|x_{n+1}-x_n\| \\ & \ +2\sup\{\|S_{n+1}z-S_nz\| : z \in K\} \\ & \ +M_2(|\alpha_{n+1}-\alpha_n|+|\beta_{n+1}-\beta_n|), \end{aligned} \tag{3.5}$$

where $M_2 = M_1(\gamma+1)$. Assume $\sum_{n=1}^{\infty} |\alpha_{n+1}-\alpha_n| < \infty$. Setting $r_n = M_2(|\alpha_{n+1} - \alpha_n| + |\beta_{n+1} - \beta_n|) + 2\sup\{\|S_{n+1}z - S_n z\| : z \in K\}$

$$\sum_{n=1}^{\infty} r_n = M_2 \sum_{n=1}^{\infty}(|\alpha_{n+1}-\alpha_n|+|\beta_{n+1}-\beta_n|)+2\sum_{n=1}^{\infty}\sup\{\|S_{n+1}z-S_n z\| : z \in K\} < \infty.$$

Therefore, it follows from Lemma 2.2, $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. Now, suppose $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$. From (3.5), we get

$$\begin{aligned}\|x_{n+2} - x_{n+1}\| \leq \quad & (1 - \alpha_{n+1}(\tau - \gamma\alpha))\|x_{n+1} - x_n\| \\ & +2\sup\{\|S_{n+1}z - S_n z\| : z \in K\} + M_2|\beta_{n+1} - \beta_n| \\ & +\alpha_{n+1}M_2|1 - \tfrac{\alpha_n}{\alpha_{n+1}}|.\end{aligned}$$

Setting $r_n = M_2|\beta_{n+1} - \beta_n| + 2\sup\{\|S_{n+1}z - S_n z\| : z \in K\}$, we have

$$\sum_{n=1}^{\infty} r_n = M_2 \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| + 2\sum_{n=1}^{\infty}\sup\{\|S_{n+1}z - S_n z\| : z \in K\} < \infty.$$

Therefore, it follows from Lemma 2.2, $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

Step 3. $\lim_{n\to\infty}\|x_n - y_n\| = 0$. Indeed, from (3.1), we obtain

$$\|x_{n+1} - y_n\| = \beta_n\|y_n - S_n y_n\|.$$

If $\lim_{n\to\infty}\beta_n = 0$, $\lim_{n\to\infty}\|x_{n+1} - y_n\| = 0$. If $\beta_n \in [0, b)$ for some $b \in (0,1)$, we have

$$\begin{aligned}\|x_{n+1} - y_n\| = \quad & \beta_n\|y_n - S_n y_n\| \\ \leq \quad & \beta_n(\|y_n - S_n x_n\| + \|S_n x_n - S_n y_n\|) \\ \leq \quad & b(\|y_n - S_n x_n\| + \|x_n - y_n\|) \\ \leq \quad & b(\|y_n - x_{n+1}\| + \|x_n - x_{n+1}\| + \|y_n - S_n x_n\|).\end{aligned}$$

Hence

$$\|x_{n+1} - y_n\| \leq \frac{b}{1-b}(\|x_n - x_{n+1}\| + \|y_n - S_n x_n\|).$$

So, by Step 2 and

$$\lim_{n\to\infty}\|y_n - S_n x_n\| = \lim_{n\to\infty}\alpha_n\|\gamma f(x_n) - \mu F(S_n x_n)\| = 0, \tag{3.6}$$

we have $\lim_{n\to\infty}\|x_{n+1} - y_n\| = 0$. This implies $\lim_{n\to\infty}\|x_n - y_n\| = 0$.

Step 4. $\lim_{n\to\infty}\|x_n - Sx_n\| = 0$. Since

$$\|x_n - S_n x_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \|y_n - S_n x_n\|,$$

it follows from Step 2, Step 3 and (3.6) that $\lim_{n\to\infty}\|x_n - S_n x_n\| = 0$. By

$$\begin{aligned}\|x_n - Sx_n\| \leq \quad & \|Sx_n - S_n x_n\| + \|S_n x_n - x_n\| \\ \leq \quad & \sup\{\|Sz - S_n z\| : z \in \{x_n\}\} + \|x_n - S_n x_n\|\end{aligned}$$

and Lemma 2.4, we have $\lim_{n\to\infty}\|x_n - Sx_n\| = 0$.

Step 5. We claim $\limsup_{n\to\infty}\langle(\gamma f - \mu F)q, y_n - q\rangle \leq 0$, where $q = P_{\bigcap_{n=1}^{\infty} Fix(S_n)}(I - \mu F + \gamma f)(q)$. Take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle(\gamma f - \mu F)q, x_n - q\rangle = \lim_{i\to\infty}\langle(\gamma f - \mu F)q, x_{n_k} - q\rangle.$$

Since $\{x_{n_k}\}$ is bounded in H, without loss of generality, we assume $x_{n_k} \rightharpoonup z \in H$. It follows from Step 4 and Lemma 2.3 that $z \in Fix(S)$. So, from (3.2), we obtain

$$\limsup_{n\to\infty}\langle(\gamma f - \mu F)q, y_n - q\rangle = \limsup_{n\to\infty}\langle(\gamma f - \mu F)q, x_n - q\rangle = \langle(\gamma f - \mu F)q, z - q)\rangle \leq 0.$$

Step 6. $\{x_n\}$ converges strongly to q. From (3.1) and Lemma 2.1, we get

$$\begin{aligned}
\|x_{n+1}-q\|^2 = & \ \|(1-\beta_n)(y_n-q)+\beta_n(S_ny_n-q)\|^2 \\
\leq & \ \|y_n-q\|^2 = \|\alpha_n\gamma f(x_n)+(I-\alpha_n\mu F)S_nx_n-q\|^2 \\
= & \ \|\alpha_n(\gamma f(x_n)-\mu F(q))+(I-\alpha_n\mu F)(S_nx_n-q)\|^2 \\
\leq & \ \|(I-\alpha_n\mu F)(S_nx_n-q)\|^2+2\alpha_n\langle\gamma f(x_n)-\mu F(q),y_n-q\rangle \\
\leq & \ (1-\alpha_n\tau)^2\|x_n-q\|^2+2\alpha_n\langle\gamma f(x_n)-\gamma f(q),y_n-q\rangle \\
& +2\alpha_n\langle\gamma f(q)-\mu F(q),y_n-q\rangle \\
\leq & \ (1-\alpha_n\tau)^2\|x_n-q\|^2+2\alpha_n\gamma\alpha\|x_n-q\|(\|y_n-x_n\|+\|x_n-q\|) \\
& +2\alpha_n\langle\gamma f(q)-\mu F(q),y_n-q\rangle \\
\leq & \ (1-2\alpha_n(\tau-\gamma\alpha))\|x_n-q\|^2+(\alpha_n\tau)^2\|x_n-q\|^2 \\
& +2\alpha_n\gamma\alpha\|x_n-q\|\|y_n-x_n\|+2\alpha_n\langle\gamma f(q)-\mu F(q),y_n-q\rangle \\
\leq & \ (1-2\alpha_n(\tau-\gamma\alpha))\|x_n-q\|^2+2\alpha_n(\tau-\gamma\alpha)\{\tfrac{(\alpha_n\tau^2)M_3^2}{2(\tau-\gamma\alpha)} \\
& +\tfrac{\gamma\alpha M_3}{\tau-\gamma\alpha}\|y_n-x_n\|+\tfrac{1}{\tau-\gamma\alpha}\langle\gamma f(q)-\mu F(q),y_n-q\rangle\} \\
= & \ (1-\delta_n)\|x_n-q\|^2+\delta_n\theta_n,
\end{aligned}$$

where $M_3=\sup\{\|x_n-q\| : n\geq 1\}$, $\delta_n=2\alpha_n(\tau-\gamma\alpha)$ and $\theta_n=\frac{(\alpha_n\tau^2)M_3^2}{2(\tau-\gamma\alpha)}+\frac{\gamma\alpha M_3}{\tau-\gamma\alpha}\|y_n-x_n\|+\frac{1}{\tau-\gamma\alpha}\langle\gamma f(q)-\mu F(q),y_n-q\rangle$. It is easy to see, $\lim_{n\to\infty}\delta_n=0$, $\sum_{n=1}^{\infty}\delta_n=\infty$ and $\limsup_{n\to\infty}\theta_n\leq 0$. Hence by Lemma 2.2, $\{x_n\}$ converges strongly to q. From Step 4, Step 6 and Lemma 2.3, we have $q\in Fix(S)$. This completes the proof. □

Taking $F=A$ (A is a strongly positive bounded linear operator on H), $\mu=1$ in Theorem 3.1, we get

Corollary 3.2. *we have $\{x_n\}$ generated by*

$$\begin{cases} y_n=\alpha_n\gamma f(x_n)+(I-\alpha_nA)S_nx_n, \\ x_{n+1}=(1-\beta_n)y_n+\beta_nS_ny_n, \quad n\geq 1, \end{cases}$$

converges strongly to $q\in Fix(S)$ which solves the variational inequality

$$\langle(A-\gamma f)q,q-z)\rangle\leq 0, \text{ for all } z\in Fix(S).$$

Taking $F=I$, $\mu=1$, $\gamma=1$ in Theorem 3.1, we get

Corollary 3.3. *we have $\{x_n\}$ generated by*

$$\begin{cases} y_n=\alpha_nf(x_n)+(1-\alpha_n)S_nx_n, \\ x_{n+1}=(1-\beta_n)y_n+\beta_nS_ny_n, \quad n\geq 1, \end{cases}$$

converges strongly to $q\in Fix(S)$ which solves the variational inequality

$$\langle(I-f)q,q-z)\rangle\leq 0, \text{ for all } z\in Fix(S).$$

Remark 3.4. Theorem 3.1 can be obtained without assumption $\tau<1$. Therefore, Theorem 3.1 is a generalization of Theorem 2.6.

Proof. We only use the assumption $\tau<1$ for finding $q\in H$ which solves the variational inequality (3.2) in Theorem 3.1. It is needed to prove Step 5 of the proof of Theorem 3.1. So, we just retrieve Step 5 of the proof of Theorem 3.1. By Theorem 2.5, there exists $q\in Fix(S)$ such that

$$\langle(\mu F-\gamma f)q,q-z)\rangle\leq 0, \text{ for all } z\in Fix(S).$$

Thus the Step 5 in the proof of Theorem 3.1 is obtained. The rest of the proof is similar to the original one. □

Remark 3.5. Corollary 3.3 is a generalization of [9, Theorem 3.2].

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