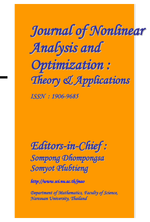
Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 4, No. 1, (2013), 99-110
ISSN : 1906-9605
http://www.math.sci.nu.ac.th



A STUDY OF NON-ATOMIC MEASURES AND INTEGRALS ON EFFECT ALGEBRAS

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ABSTRACT. The present paper deals with the study of superior variation m^+, inferior variation m^- and total variation $|m|$ of an extended real-valued function m defined on an effect algebra L. Various properties in the context of functions m^+, m^- and $|m|$ are also established. Using the notion of an atom of a real-valued function, we have proved Intermediate value theorem for a non-atomic function m defined on a D-lattice L under suitable conditions. Finally, the notion of the integral for a bounded, real valued function with respect to a measure on effect algebras with Reisz decomposition property is introduced and studied.

KEYWORDS : Effect algebra; Superior variation; Inferior variation; Total variation; Jordan type decomposition theorem; m-Atoms; Non-atomic measure; Intermediate value theorem; Reisz decomposition property; μ-Integrable
AMS Subject Classification: 06A11 06C15 28A12 28E99

1. INTRODUCTION

If a quantum mechanical system $\mathcal{F}$ is represented in the usual way by a Hilbert space $\mathcal{H}$, then a self adjoint operator A on $\mathcal{H}$ such that $0 \leq A \leq I$ corresponds to an effect for $\mathcal{F}$ [19, 20, 29]. Effects are of significance in representing unsharp measurements or observations on the system $\mathcal{F}$ [4], and effect valued measures play an important role in stochastic quantum mechanics [1, 30]. As a consequence, there have been a number of recent efforts to establish appropriate axioms for logics, algebras, or posets suggested by or based on effects [13, 14]

In 1992, Kôpka defined D-posets of fuzzy sets in [13], which is closed under the formations of differences of fuzzy sets, while studying the axiomatical systems of fuzzy sets. A generalization of such structures of fuzzy sets to an abstract partially ordered set, where the basic operation is the difference, yields a very general and, at the same time, a very simple structure called a D-poset. A common generalization of orthomodular lattices and MV-algebras is termed as lattice ordered effect algebras introduced by Bennett and Foulis [4, 5] in 1994, while working on quantum

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Article history : Received 28 March 2012. Accepted 5 October 2012.

mechanical systems. Such structures are being frequently used because of their wide range of applications in quantum physics, mathematical economics and fuzzy theory. For a list of nice examples of effect algebras, we refer to [8] and for some of its properties we refer also [5] and [6]. The equivalence of D-posets and effect algebras is proved by Foulis and Bennett [4] and independently by Pulmannová [25].

The decomposability of a vector measure was first studied by Rickart in 1943 [26], where he established a Lebesgue decomposition theorem for the class of “strongly bounded" additive measures. This result was later re-established (although it was not realized at that time) by Uhl. Jr. [30], who also presented a Yosida-Hewitt decomposition theorem for “strongly bounded" measures. Several Jordan type decomposition theorems are exhibited by Diestal and Faires in [7]. Afterwards, Faires and Morrison [9] exposed conditions on a vector valued measures that ensure vector valued Jordan type decomposition theorem to hold. A Jordan type decomposition theorem for vector measures, defined on an algebra of of sets, with values in an order complete Banach lattice is proved by Schmidt [27]. Upto slight modification, the result of [28]. extends to the case where domain of the vector measure is a ring of sets. It is also possible to give a common approach to vector measures on a Boolean ring and linear operators on a vector lattice. A first step in this direction was done in [27], where real-valued case was studied. The method presented there is based on a common abstraction of Boolean rings and lattice ordered groups. This approach can be refined and fitted to the vector valued case, and it then yields results of [7, 11] on Jordan decomposition without appeal to regularity of representing linear operators. The notion of non-atomic measures and their properties are studied by [15, 16, 17, 18, 21] and the references therein.

Aumann [2] introduced the concept of integral of a set-valued function which have many applications in mathematical economics, theory of control and many other fields. Different approaches has been use to extend and generalize the Integral theory. In the field of set-valued integrals, another approach was done by many authors using the Choquet integral or the Sugeno fuzzy integral (see [11, 12, 21, 24] and the references therein). Gould [11] investigated an integral of a real function with respect to an additive measure taking values in a Banach space X.

The objective of the present paper is study the notion of the non-atomic measures and integrals on effect algebras. The notion of non-atomic measures is used to establish an Intermediate value theorem on effect algebras. Moreover, the notion of integrals is introduced and studied with some of the basic natural properties of the integrals on effect algebras.

The paper is organized as follows: Section 2 contains prerequisites and basic results on an effect algebra L. The notions of superior variation m^+, inferior variation m^- and total variation $|m|$ [17] of an extended real-valued function m defined on L are studied elaborately in Section 3, followed by various properties in the context of functions m^+, m^- and $|m|$. Using the notion of an atom of a real-valued measure m [15, 17], we have proved the equivalence of the following: (i) m^+ and m^- are non-atomic, (ii) $|m|$ is non-atomic, (iii) m is non-atomic. In Section 4, we have proved Intermediate value theorem for a locally bounded real-valued σ-additive, non-atomic function m defined on a σ-continuous, σ-complete D-lattice L. Section 5 is concentrated on the objective to introduce the notion of the integral for a bounded, real valued function with respect to a measure on effect algebras with Reisz decomposition property (see [12, 22]).

2. PRELIMINARIES AND BASIC RESULTS

First of all, we shall give some preliminaries and basic results from effect algebras, which can be found in [3, 4, 5, 6, 22].

An *effect algebra* $(L; \oplus, 0, 1)$ is a structure consisting of a set L, two special elements 0 and 1, and a partially defined binary operation $\oplus$ on $L \times L$ satisfying the following conditions for $a, b, c \in L$:

(1) $a \oplus b = b \oplus a$, if $a \oplus b$ is defined;

(2) $a \oplus (b \oplus c) = (a \oplus b) \oplus c$, if one side is defined;

(3) for every $a \in L$, there exists a unique $b \in L$ such that $a \oplus b = 1$ (we put $a^{\perp} = b$);

(4) if $a \oplus 1$ is defined, then $a = 0$.

For brevity, we denote an effect algbera $(L; \oplus, 0, 1)$ by L. In an effect algebra L, a dual operation $\ominus$ to $\oplus$ can be defined as follows: $a \ominus c$ exists and equals b if and only if $b \oplus c$ exists and equals a. We say that two elements $a, b \in L$ are *orthogonal*, and we write $a \perp b$, if $a \oplus b$ exists. If $a \oplus b = 1$, then b is called *orthocomplement* of a and write $b = a^{\perp}$. It is obvious that $1^{\perp} = 0$, $(a^{\perp})^{\perp} = a$, $a \perp 0$ and $a \oplus 0 = a$, for all $a \in L$. Also, for $a, b \in L$, we define $a \leq b$ if there exists $c \in L$ such that $a \perp c$ and $a \oplus c = b$. It may be proved that $\leq$ is a partial ordering on L and $0 \leq a \leq 1$; $a \leq b \Leftrightarrow b^{\perp} \leq a^{\perp}$, and $a \leq b^{\perp} \Leftrightarrow a \perp b$ for $a, b \in L$. If $a \leq b$, the element $c \in L$ such that $c \perp a$ and $a \oplus c = b$ is unique, and satisfies the condition $c = (a \oplus b^{\perp})^{\perp}$ (we put $c = b \ominus a$).

In a natural way, the sum of more than two elements is obtained: If $a_1, a_2 \ldots, a_n \in L$, we inductively define $a_1 \oplus a_2 \oplus \ldots \oplus a_n = (a_1 \oplus \ldots \oplus a_{n-1}) \oplus a_n$, provided that the right hand side exists. The definition is independent on permutations of the elements. We say that a finite subset $\{a_1, a_2, \ldots, a_n\}$ of L is *orthogonal* if $a_1 \oplus a_2 \oplus \ldots \oplus a_n$ exists. For a sequence $\{a_n\}$, we say that it is *orthogonal* if, for every n, $\bigoplus_{i \leqslant n} a_i$ exists. If, moreover, $\sup_n \bigoplus_{i \leqslant n} a_i$ exists, the *sum* $\bigoplus_{n \in \mathbb{N}} a_n$ of an orthogonal sequence $\{a_n\}$ in L is defined as $\sup \bigoplus_{i \leqslant n} a_i$; we denote by $\mathbb{N}$ the set of all natural numbers and by $\mathbb{R}$ the set of all real numbers. An effect algebra L is called a *σ-complete effect algebra*, if every orthogonal sequence in L has its sum. If $(L, \leq)$ is a lattice, we say that effect algebra is a *lattice ordered effect algebra* (or a *D-lattice*). The notion of σ-continuity of a D-lattice is, as usual, expressed in terms of monotone sequences: we write $a_n \uparrow a$ (respectively, $a_n \downarrow a$) whenever $\{a_n\}$ is an increasing sequence in L and $a = \sup_n a_n$ (respectively, $\{a_n\}$ is a decreasing sequence and $a = \inf_n a_n$). The lattice $(L, \leq)$ is said to be *σ-continuous* if $a_n \uparrow a$ implies $a_n \wedge b \uparrow a \wedge b$ (or equivalently, $a_n \downarrow a$ implies $a_n \vee b \downarrow a \vee b$) for all $b \in L$.

2.1. We say that an effect algebra L satisfies the *Riesz decomposition property*, *RDP* for short, if for all $a, b_1, b_2 \in L$ $a \leq b_1 \oplus b_2$ implies that there exists two elements $a_1, a_2 \in L$ with $a_1 \leq b_1$ and $a_2 \leq b_2$ such that $a = a_1 \oplus a_2$. L has RDP if and only if, for $x_1, x_2, y_1, y_2 \in L$ such that $x_1 \oplus x_2 = y_1 \oplus y_2$ implies there exists four elements $c_{11}, c_{12}, c_{21}, c_{22} \in L$ such that $x_1 = c_{11} \oplus c_{12}$, $x_2 = c_{21} \oplus c_{22}$, $y_1 = c_{11} \oplus c_{21}$ and $y_2 = c_{12} \oplus c_{22}$.

2.2. A finite sequence $\mathcal{A} = \{a_i\}_{i=1}^{n}$ of nonzero elements of an effect algebra L is called a *partition* of unity of L if $a_1 \oplus a_2 \oplus \cdots \oplus a_n = 1$. A partition $\mathcal{B} = \{b_j\}_{j=1}^{m}$ is called a *refinement* of the partition $\mathcal{A}$ denoted by $\mathcal{A} \prec \mathcal{B}$, if for any element a_i ($i = 1, 2, \ldots, n$) there is a subset $\alpha_i \subseteq \{1, 2, \ldots, m\}$ such that $a_i = \bigoplus_{j \in \alpha_i} b_j$ and $\bigcup_{i=1}^{m} \alpha_i = \{1, 2, \ldots, m\}$ and $\alpha_i \cap \alpha_k = \phi$ for $i \neq k$.

Let $\mathcal{A} = \{a_i\}_{i=1}^{n}$ and $\mathcal{B} = \{b_j\}_{j=1}^{m}$ be two partitions of unity in an effect algebra L with RDP. Due to RDP, there is a *Riesz refinement* (*or joint refinement*) $\mathcal{C} = \{c_{ij} :$

$1 \leq i \leq n, 1 \leq j \leq m\}$ of $\{a_i\}_{i=1}^n$ and $\{b_j\}_{j=1}^m$ such that, for all $1 \leq i \leq n$ and all $1 \leq j \leq m$. We have $a_i = c_{i1} \oplus \cdots \oplus c_{im}$, $b_j = c_{1j} \oplus \cdots \oplus c_{nj}$. In this case $\mathcal{C}$ is a partition of unity of L, such that $\mathcal{A} \prec \mathcal{C}$ and $\mathcal{B} \prec \mathcal{C}$. Also the family $(\mathfrak{P}, \prec)$ is a directed set, where $\mathfrak{P}$ is the set of all partitions of unity of L, $\prec$ is the order relation on $\mathfrak{P}$.

2.3. A function $\mu : L \to \mathbb{R}$ is said to be a *measure* if $\mu(a \oplus b) = \mu(a) + \mu(b)$, for every $a \oplus b \in L$. If μ is measure then $\mu(0) = 0$. If range of μ is $[0, \infty)$, then μ is monotone. A measure $\mu : L \to \mathbb{R}$ is said to be of *finite variation* if $\sup\{|\mu(a)| : a \in L\} < \infty$.

Let us recall the following results:

2.4. Assume that a, b, c are elements of an effect algebra L.

(i) If $a \leqslant b$, then $b = a \oplus (b \ominus a)$.

(ii) If $a \perp b$, then $a \leq a \oplus b$ and $(a \oplus b) \ominus a = b$.

(iii) If $a \leqslant b \leqslant c$, then $(b \ominus a) \leq (c \ominus a)$.

(v) If $a \leq b \leq c$, then $a \oplus (c \ominus b) = c \ominus (b \ominus a)$ and $(c \ominus b) \oplus (b \ominus a) = (c \ominus a)$.

(vi) If $a \leq b \leq c$, then $(c \ominus b) \leq (c \ominus a)$ and $(c \ominus a) \ominus (c \ominus b) = (b \ominus a)$.

(vii) If $a \leq b \leq c$, then $(b \ominus a) \leq (c \ominus a)$ and $(c \ominus a) \ominus (b \ominus a) = (c \ominus b)$.

(viii) If $a \leq b \leq c$, then $a \perp (c \ominus b)$ and $a \oplus (c \ominus b) = c \ominus (b \ominus a)$.

(ix) If $a \leq b' \leq c'$, then $a \oplus (b \ominus c) = (a \oplus b) \ominus c$.

(x) If $a \perp b$ and $(a \oplus b) \leq c$, then $c \ominus (a \oplus b) = (c \ominus a) \ominus b = (c \ominus b) \ominus a$.

2.5. Let L be a σ-complete effect algebra. If $\{a_n\}$ is an increasing (respectively, decreasing) sequence, then $\sup_n a_n$ (respectively, $\inf_n a_n$) exists.

2.6. A function m defined on an effect algebra L with values in $\mathbb{R}$ is called a *measure* on L, if $a, b \in L$, $a \perp b$ implies $m(a \oplus b) = m(a) + m(b)$. It is clear that m is a measure if and only if $b \leq a$ implies $m(a) = m(b) + m(a \ominus b)$. Obviously, if m is a measure, then: (i) $m(0) = 0$; (ii) if $\beta \neq 0$ is a finite number, then βm is also a measure. We say that m is *σ-additive*, if for every orthogonal sequence $\{a_n\}$ in L such that $\bigoplus_n a_n$ exists, $m(\bigoplus_n a_n) = \sum_{n=1}^{\infty} m(a_n)$.

2.7. A function m defined on a D-lattice L with values in $\mathbb{R}$, is called *modular*, if $m(a \vee b) + m(a \wedge b) = m(a) + m(b)$ for $a, b \in L$.

2.8. A function m defined on an effect algebra L with values in $\mathbb{R}$, is called *locally bounded* if, for any $a \in L$, $\sup\{m(b) : b \leqslant a, b \in L\}$ exists.

3. NON-ATOMIC MEASURES ON EFFECT ALGEBRAS

Definition 3.1. [17] Let m be an extended real-valued function defined on an effect algebra L, that is, $m : L \to [-\infty, \infty]$, with $m(0) = 0$. Then for $a \in L$,

(i) *superior variation* of m is defined by

$$m^+(a) = \sup\{m(b) : b \leqslant a, b \in L\};$$

(ii) *inferior variation* of m is defined by

$$\begin{aligned} m^-(a) &= -\inf\{m(b) : b \leqslant a, b \in L\} \\ &= \sup\{-m(b) : b \leqslant a, b \in L\}; \end{aligned}$$

(iii) *total variation* of m is defined by

$$|m| = m^+ + m^-.$$

Remark 3.2. *(i)* $0 \leq m^+(a) \leq \infty$, $0 \leq m^-(a) \leq \infty$, $0 \leq |m|(a) \leq \infty$, $a \in L$;

(ii) $m^+(0) = 0 = m^-(0)$, $|m|(0) = 0$;

(iii) $m^- = (-m)^+$, $m^+ = (-m)^-$;

(iv) $-m^-(a) \leqslant m(a) \leqslant m^+(a)$, $|m(a)| \leq |m|(a)$, $a \in L$.

Theorem 3.3. *If m is a locally bounded real-valued measure defined on an effect algebra L, then m can be written as*

$$m = m^+ - m^-.$$

Proof Let $\varepsilon > 0$. Let $a \in L$. Then there exists $b \in L$ such that $b \leqslant a$ and

$$m^+(a) - \varepsilon < m(b). \tag{3.1}$$

Since $a \ominus b \leq a$, we have

$$-m(a \ominus b) \leq m^-(a). \tag{3.2}$$

From (3.1) and (3.2), we get

$$m^+(a) - \varepsilon - m^-(a) < m(b) + m(a \ominus b),$$

which yields that

$$m^+(a) - m^-(a) - \varepsilon < m(a). \tag{3.3}$$

Letting $\varepsilon \to 0$, we obtain

$$m^+(a) - m^-(a) \leq m(a). \tag{3.4}$$

Further, since (3.4) is true for any m, with the aid of Remark 3.2(iii), we have

$$m^+(a) - m^-(a) \geq m(a). \tag{3.5}$$

Thus (3.4) and (3.5) yields that

$$m(a) = m^+(a) - m^-(a),$$

or $m = m^+ - m^-$. □

Theorem 3.4. *Let m be a real-valued modular measure defined on a D-lattice L. Then m^+ and m^- are measures (and hence $|m|$ is also a measure).*

Proof Firstly, let us consider about m^+. We have proved in [16], that for $a, b \in L$ with $a \perp b$,

$$m^+(a \oplus b) \leq m^+(a) + m^+(b). \tag{3.6}$$

By Definition 3.1 (i), (ii), there are sequences $\{a_n\}$ and $\{b_n\}$ of elements from L such that $a_n \leq a$, $b_n \leq b$ with

$$m(a_n) \to m^+(a), \;\; m(b_n) \to m^+(b). \tag{3.7}$$

Obviously, $a_n \perp b_n$ for each n. Therefore, from $m(a_n \oplus b_n) = m(a_n) + m(b_n)$, we have $m(a_n \oplus b_n) \to m^+(a) + m^+(b)$.

Further, $a_n \oplus b_n \leq a \oplus b$ yields that

$$m^+(a \oplus b) \geq m^+(a) + m^+(b). \tag{3.8}$$

From (3.6) and (3.8), we get

$$m^+(a \oplus b) = m^+(a) + m^+(b),$$

that is, m^+ is a measure.

By similar argument, we can show that m^- is a measure. From Definition 3.1(iii), $|m|$ is a measure.

Definition 3.5. An extended real-valued function m defined on an effect algebra L is called *continuous from below* (respectively, *continuous from above*), if $a, a_n \in L$, $a_n \uparrow a$, $n \in \mathbb{N} \Rightarrow m(a) = \lim_{n \longrightarrow \infty} m(a_n)$ (respectively, if $a, a_n \in L$, $a_n \downarrow a$, $n \in \mathbb{N}$ and $m(a_1) < \infty \Rightarrow m(a) = \lim_{n \longrightarrow \infty} m(a_n)$).

Proposition 3.6. *Let $m : L \to \mathbb{R}$ be a measure. Then the following assertions are equivalent:*

(i) m is σ-additive.

(ii) m is continuous from below.

(iii) m is continuous from above.

(iv) $a_n \downarrow 0$ implies $\lim_{n\to\infty} m(a_n) = 0$.

Theorem 3.7. *If m is a locally bounded real-valued σ-additive function defined on a σ-continuous D-lattice L, then m^+, m^- and $|m|$ are also σ-additive.*

Proof Let $a_n \uparrow a,\ a, a_n \in L$. Then $m^+(a_n) \leq m^+(a)$, for every n. Thus the increasing sequence $\{m^+(a_n)\}$ converges to a limit l, say, where $l \leq m^+(a)$.

For any element $b \in L$, $b \leq a$,

$$m(b \wedge a_n) \leq m^+(b \wedge a_n) \leq m^+(a_n);$$

also, from the σ-additivity of m,

$$m(b \wedge a_n) \to m(b).$$

Hence,

$$m(b) \leq l.$$

As $b \in L$ is arbitrary, we get

$$m^+(a) \leq l.$$

It follows that $m^+(a) = l$, that is, $m^+(a_n) \to m^+(a)$.

Further, since m^+ is a measure, in view of Proposition 3.6, m^+ is σ-additive.

The σ-additivity of m^- and $|m|$ are obvious.

Theorem 3.8. *If m is a locally bounded real-valued measure defined on an effect algebra L, then m can be written as*

$$m = m^+ - m^-.$$

Further, if m is a real-valued modular measure defined on a lattice effect algebra L, then the decomposed parts m^+ and m^- are measures on L (and hence $|m|$ is also a measure on L). Moreover, if m is a locally bounded real-valued σ-additive function defined on a σ-continuous D-lattice L, then the decomposed parts m^+, m^-, and $|m|$ are also σ-additive.

4. INTERMEDIATE VALUE THEOREM

Let m be a real-valued function defined on an effect algebra L. Firstly, we shall recall the notion of an atom of a measure m defined on an effect algebra L, which has been studied in [15, 17].

Definition 4.1. An element $a \in L$ with $m(a) \neq 0$ is called an *atom of* m (or an m-*atom*), if for $a, b \in L$ with $b \leq a$,

(i) $m(b) = 0$ (that is, $a =_m 0$) or

(ii) $m(a) = m(b)$.

In case there are no atoms of m in L, m is called *non-atomic* on L.

Theorem 4.2. *Let m be a locally bounded real-valued measure defined on an effect algebra L. Then the following conditions are equivalent:*
(i) m^+ *and* m^- *are non-atomic.*
(ii) $|m|$ *is non-atomic.*
(iii) m *is non-atomic.*

Proof $(i) \Rightarrow (ii)$: Let $a \in L$ be a $|m|$-atom. Let $b \leq a$, $b \in L$ with $m^+(b) \neq 0$. Obviously, $|m|(b) \neq 0$ and hence $|m|(a) = |m|(b)$, which yields that $a \in L$ is an m^+-atom.

$(ii) \Rightarrow (iii)$: See proof of Theorem 5.5 [17].

$(iii) \Rightarrow (i)$: Let $a \in L$ is an m^+-atom. Let $b \leq a$, $b \in L$ with $m(b) \neq 0$. Obviously, $m^+(b) \neq 0$ and hence $m^+(a) = m^+(b)$, which yields that

$$m(a) \leq m(b). \tag{4.1}$$

From (4.1) and Theorem 3.8, we have

$$m^+(a) - m^-(a) \leq m(b). \tag{4.2}$$

Replacing m by $-m$ in (4.2), we get

$$m(a) \geq m(b). \tag{4.3}$$

From (4.1) and (4.3), $a \in L$ is an m-atom.

Theorem 4.3. *Let m be a $[0, \infty)$-valued σ-additive function defined on a σ-complete effect algebra L. Then m is non-atomic on L if and only if for a given element $a \in L$ with $m(a) > 0$ and $\varepsilon > 0$, there exists $b \in L$, $b \leq a$, such that $0 < m(b) < \varepsilon$.*

Proof The *if* part: Obvious.

The *only if* part: Suppose the contrary and choose an element $a \in L$ with $m(a) > 0$ and $t_0 > 0$, for which $m(b) \geqslant t_0$ holds if $b \leqslant a$, $b \in L$ and $m(b) > 0$. Define

$$t_1 = \inf\{m(b) : b \in L, b \leqslant a, m(b) > 0\}.$$

Then obviously $0 < t_0 \leqslant t_1$. Take $a_1 \leqslant a$, $a_1 \in L$ with $t_1 \leqslant m(a_1) < t_1 + 1$ and setting

$$t_2 = \inf\{m(b) : b \in L, b \leqslant a_1, m(b) > 0\}.$$

Choose $a_2 \leqslant a_1$ with $t_2 \leqslant m(a_2) < t_2 + \frac{1}{2}$. Continuing the process in the same manner, we obtain sequences $\{t_n\}$ and $\{a_n\}$ such that $t_0 \leqslant t_1 \leqslant t_2 \leqslant \ldots \leqslant m(a)$ and $a \geqslant a_1 \geqslant a_2 \geqslant \ldots$ with

$$t_n \leqslant m(a_n) < t_n + \frac{1}{2^n},$$

for all n. Using 2.2, put $a_0 = \bigwedge_{n=1}^{\infty} a_n$. Clearly, in view of Proposition 3.6, we have $m(a_0) = \lim_{n \longrightarrow \infty} m(a_n) = \lim_{n \longrightarrow \infty} t_n > 0$. Let $b \leqslant a_0$ with $m(b) > 0$. Then $\mu(a_0) \geqslant \mu(b) \geqslant t_n$, for any n and hence $\mu(b) = \mu(a_0)$. This gives that $a_0 \in L$ is an atom of m, a contradiction.

Theorem 4.4. *Let m be a $[0, \infty)$-valued σ-additive function defined on a σ-complete effect algebra L. If m is non-atomic on L, then m takes every value between 0 and $m(1)$.*

Proof Let $0 < t < m(1)$. According to Theorem 4.3, there are elements $c \in L$ such that $0 < m(c) < t$. Let

$$s_1 = \sup\{m(c) : c \in L, m(c) \leqslant t\}.$$

(Obviously $0 < s_1 \leqslant t$). Then there exists an element $c_1 \in L$ such that $\frac{s_1}{2} < m(c_1) \leqslant s_1$. Let

$$s_2 = \sup\{m(c) : c \in L, c_1 \leqslant c, m(c) \leqslant t\}.$$

Then there exists an element $c_2 \in L$ such that $c_2 \geqslant c_1$ and $s_2 - \frac{s_1}{2^2} < m(c_2) \leqslant s_2$. Continue this construction inductively to obtain

$$s_n = \sup\{m(c) : c \in L, c_{n-1} \leqslant c, m(c) \leqslant t\},$$

and then there exists $c_n \geqslant c_{n-1}$, $c_n \in L$ such that

$$s_n - \frac{s_1}{2^n} < m(c_n) \leqslant s_n.$$

It is clear that $\{s_n\}$ is a decreasing sequence and $\{c_n\}$ is an increasing sequence of elements in L such that $d = \bigvee_{n=1}^{\infty} c_n \in L$ (using 2.5) and therefore, in view of Proposition 3.6, we get $\lim_{n\longrightarrow\infty} s_n = \lim_{n\longrightarrow\infty} \mu(c_n) = \mu(d)$. Therefore $\mu(d) = \lim_{n\longrightarrow\infty} s_n = s$ (let). Clearly $s \leqslant t$. Now we claim that $s = t$. For, otherwise, let us suppose that $s < t$. Since $0 < t < \mu(1)$, we get $\mu(1 \ominus d) > 0$, $d \in L$ and therefore, by Theorem 4.3, we obtain an element b of L such that $b \leqslant (1 \ominus d)$ and $s < \mu(d \oplus b) < t$. But then $d \oplus b \geqslant c_{n-1}$, for all $n > 1$, which yields that $\mu(d \oplus b) \leqslant s_n$, for all n. This will further imply that $\mu(d \oplus b) \leqslant s$, a contradiction. Thus $\mu(d) = t$ as required.

Theorem 4.5. (*Intermediate value theorem*). *Let m be a locally bounded real-valued σ-additive function defined on a σ-continuous, σ-complete D-lattice L. If m is non-atomic on L, then m takes every value between $-m^-(1)$ and $m^+(1)$.*

Proof Follows from Theorem 3.7, Theorem 4.2 and Theorem 4.4.

5. GOULD TYPE INTEGRAL ON AN EFFECT ALGEBRA

In this section, we introduce and study the notion of Gould type integral with respect to a measure defined on an effect algebra. The Gould type integral was intensively studied for different types of set (multi)functions: vector valued measures [11], measures [12] monotone set multifunctions (called fuzzy multimeasures) [24]. From now onwards, let μ be a measure defined on an effect algebra L with RDP, which is not identically zero and let $f : L \to \mathbb{R}$ be a real valued bounded function.

Definition 5.1. [12] For a measure μ on L, define

$$\overline{\mu}(a) = \sup\{\sum_{i=1}^{n} \mid \mu(a_i) \mid\},$$

for every $a \in L$, where the supremum is extended over all finite partitions $\{a_i\}_{i=1}^n$ of a. We define $\widetilde{\mu}$ as

$$\widetilde{\mu}(a) = \inf\{\overline{\mu}(b) : a \leq b,\ b \in L\},$$

for every $a \in L$. It may be observed that $\overline{\mu}$ is a finitely additive monotone function on L and $\widetilde{\mu}(a) = \overline{\mu}(a)$, for every $a \in L$.
A property (M) is said to be μ-*almost every where* (μ-$a.e.$ in brief), if the property (M) is valid otherwise the set $\{a \in L : \widetilde{\mu}(a) = 0\}$. We shall assume $\widetilde{\mu}(1) < \infty$.

Definition 5.2. [12] Define $osc(f, a) = \sup_{x,y \leq a} \mid f(x) - f(y) \mid$, where $a \in L$. We observe that:

(1) $a \leq b \Rightarrow osc(f, a) \leq osc(f, b)$, for $a, b \in L$.
(2) $osc(f + g, a) \leq osc(f, a) + osc(g, a)$, for $a \in L$.
(3) $osc(\alpha f, a) =\mid \alpha \mid osc(f, a)$, for $a \in L$ and $\alpha \in \mathbb{R}$.

The function f is said to be $\widetilde{\mu}$-*measurable* on L if for every $\varepsilon > 0$ there exists a partition $\mathcal{A}_\varepsilon = \{a_i\}_{i=0}^n$ of unity of L such that

(1) $\widetilde{\mu}(a_0) < \varepsilon$,
(2) $\sup_{x,y \leq a_i} \mid f(x) - f(y) \mid = osc(f, a_i) < \varepsilon$, for every $i = 1, 2, \ldots, n$.
Such a partition $\mathcal{A}_\varepsilon$ is called an *ε-partition* of unity of L.
It is easy to see that if f and g are $\widetilde{\mu}$-measurable on L, then $f + g$ is $\widetilde{\mu}$-measurable, αf is $\widetilde{\mu}$-measurable for every $\alpha \in \mathbb{R}$ and $f + c$ is $\widetilde{\mu}$-measurable for every constant real number c.

Definition 5.3. [12] Let $\sigma(\mathcal{A}, t_i, \mu) \equiv \sigma(\mathcal{A}) = \sum_{i=1}^{n} f(t_i)\mu(a_i)$ for every partition $\mathcal{A}$ = $\{a_i\}_{i=1}^{n}$ of unity of L and $t_i \leq a_i, t_i \in L$, $i = 1, 2, \ldots, n$. The function f is said to be *μ-integrable* on L if there exists a $I \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists a partition $\mathcal{A}_\varepsilon$ of unity of L so that for every partition $\mathcal{A}$ = $\{a_i\}_{i=1}^{n} \in \mathfrak{P}$ with $\mathcal{A}_\varepsilon \prec \mathcal{A}$ and every choice of $t_i \leq a_i, t_i \in L$, $i = 1, 2, \ldots, n$ we have $d(\sigma(\mathcal{A}), x) < \varepsilon$ (here d is the usual metric on $\mathbb{R}$). In this case I is called the integral of f in L and is denoted by $\int_L f d\mu$. That is, the net $\{\sigma(\mathcal{A})\}_{\mathcal{A} \in (\mathfrak{P}, \prec)}$ is convergent in $(\mathbb{R}, d)$. Obviously, if $\mathcal{A}_\varepsilon$ exists, the integral is unique.

Example 5.4. Let $M_k = \{0, \frac{1}{k}, \frac{2}{k}, \ldots, \frac{k}{k}\}$, $k \geq 1$, be a finite MV-algebra. Then $s(\frac{1}{k}) = \frac{1}{k}$ is the measure on M_k and $\mathcal{A}$ = $\{\frac{1}{k}, \frac{1}{k}, \ldots, \frac{1}{k}\}$ is the finest refinement of unity in M_k. Let $f : M_k \to \mathbb{R}$ is a bounded real valued function defined as $f(x) = c$, for every $x \in L$, where c is a real constant. Then $\sum_n f(\frac{1}{k})s(\frac{1}{k}) = c$ and hence $\int_L f d\mu = c$

Theorem 5.5. *If μ is nonnegative and $f = 0$ μ-almost everywhere, then f is $\widetilde{\mu}$-measurable, μ-integrable on L and $\int_L f d\mu = 0$.*

Proof Since f is a bounded function on L, there exists $M > 0$ so that $\mid f(a) \mid \leq M$ for every $a \in L$. Let $A = \{a \in L : f(a) \neq 0\}$. Then $\widetilde{\mu}(a) = 0$ for every $a \in A$, which implies that, for every $\varepsilon > 0$ there exists $b_\varepsilon \in L$, so that $a \leq b_\varepsilon$ and $\overline{\mu}(b_\varepsilon) < \frac{\varepsilon}{M}$. Let us consider $\mathcal{A}_\varepsilon = \{b_\varepsilon, b_\varepsilon^\perp\}$ which is a partition of unity of L. Since $\widetilde{\mu}(b_\varepsilon) = \overline{\mu}(b_\varepsilon) < \frac{\varepsilon}{M}$ and $\mathrm{osc}(f, b_\varepsilon^\perp) = \sup_{x,y \leq b_\varepsilon} \mid f(x) - f(y) \mid = 0$, we get that f is $\widetilde{\mu}$- measurable.
Let us take an arbitrary partition $\mathcal{B}$ = $\{b_j\}_{i=1}^{m}$ of unity of L, such that $\mathcal{A}_\varepsilon \prec \mathcal{B}$. Let $t_i \leq b_i, t_i \in L$, $i = 1, 2, \ldots, m$, be chosen arbitrary. We may suppose that $b_1 \oplus b_2 \oplus \ldots, \oplus b_k = b_\varepsilon$ and $b_{k+1} \oplus b_{k+2} \oplus \cdots \oplus b_m = b_\varepsilon^\perp$. Then

$$\begin{aligned}
\sigma(\mathcal{A}) =\mid \sum_{i=1}^{m} f(t_i)\mu(b_i) \mid &\leq \sum_{i=1}^{k} \mid f(t_i)\mu(b_i) \mid + \sum_{i=k+1}^{m} \mid f(t_i)\mu(b_i) \mid \\
&= \sum_{i=1}^{k} \mid f(t_i)\mu(b_i) \mid \\
&\leq M \cdot \sum_{i=1}^{k} \mid \mu(b_i) \mid \\
&\leq M \cdot \overline{\mu}(b_\varepsilon) \leq M \cdot \frac{\varepsilon}{M} = \varepsilon.
\end{aligned}$$

Hence f is μ-integrable on L and $\int_L f d\mu = 0$.

Theorem 5.6. *If f is μ-integrable on L and $\alpha \in \mathbb{R}$, then*
(1) αf is μ-integrable on L and

$$\int_L \alpha f d\mu = \alpha \int_L f d\mu,$$

(2) f is $\alpha\mu$-integrable on L and

$$\int_L f d(\alpha\mu) = \alpha \int_L f d\mu.$$

Proof The case $\alpha = 0$ is trivial. Let $\alpha \neq 0$. Because f is μ-integrable on L, for every $\varepsilon > 0$ there exists a partition $\mathcal{A}_\varepsilon \in \mathfrak{P}$, so that for every partition $\mathcal{A} = \{a_i\}_{i=1}^n \in \mathfrak{P}$ with $\mathcal{A}_\varepsilon \prec \mathcal{A}$ and for every $t_i \leq a_i, t_i \in L$, $i = 1, 2, \ldots, n$ we have

$$d(\sum_{i=1}^{n} f(t_i)\mu(a_i), \int_L f d\mu) < \frac{\varepsilon}{\mid \alpha \mid}.$$

Then

$$\begin{aligned} d(\sum_{i=1}^{n} \alpha f(t_i)\mu(a_i), \alpha \int_L f d\mu) &=\mid \alpha \mid \cdot d(\sum_{i=1}^{n} f(t_i)\mu(a_i), \int_L f d\mu), \\ &<\mid \alpha \mid \cdot \frac{\varepsilon}{\mid \alpha \mid} = \varepsilon, \end{aligned}$$

that is, αf is μ-integrable on L and $\int_L \alpha f d\mu = \alpha \int_L f d\mu$.

(2) The function $\mu : L \to \mathbb{R}$ defined by $(\alpha\mu)(a) = \alpha.\mu(a)$, for every $a \in L$ is a measure of finite variation and hence the theorem.

Theorem 5.7. *Let $f, g : L \to \mathbb{R}$ are two bounded μ-integrable functions on L. Then $f + g$ is also μ-integrable on L and*

$$\int_L (f + g) d\mu = \int_L f d\mu + \int_L g d\mu.$$

Proof Since f is μ-integrable, then for every $\varepsilon > 0$, there exists $\mathcal{A}_1 \in \mathfrak{P}$ so that for every $\mathcal{B}_1 \in \mathfrak{P}$, $\mathcal{B}_1 = \{a_i\}_{i=1}^n$ with $\mathcal{A}_1 \prec \mathcal{B}_1$ and for every $t_i \leq a_i, t_i \in L$, $i = 1, 2, \ldots, n$, we have

$$d(\sum_{i=1}^{n} f(t_i)\mu(a_i), \int_L f d\mu) < \frac{\varepsilon}{2}.$$

Similarly, g is also μ-integrable so there is a partition $\mathcal{A}_2 \in \mathfrak{P}$ such that for every $\mathcal{B}_2 \in \mathfrak{P}$, $\mathcal{B}_2 = \{b_j\}_{j=1}^m$ with $\mathcal{A}_2 \prec \mathcal{B}_2$ and every $s_j \leq b_j, s_j \in L$, $j = 1, 2, \ldots, m$, we have

$$d(\sum_{j=1}^{m} f(s_j)\mu(b_j), \int_L g d\mu) < \frac{\varepsilon}{2}.$$

Let $\mathcal{A}_0$ be a joint refinement of $\mathcal{A}_1$ and $\mathcal{A}_2$. Then for any partition $\mathcal{C} = \{c_k\}_{k=1}^p \in \mathfrak{P}$ with $\mathcal{A}_0 \prec \mathcal{C}$, we get that for every $k = 1, 2, \ldots, p$, there exists $i_k = 1, 2, \ldots, n$ and $j_k = 1, 2, \ldots, m$, so that $c_k \leq a_{i_k}, c_k \in L$ and $c_k \leq b_{j_k}, c_k \in L$. We have to prove that $d(\sum_{k=1}^p (f+g)(r_k)\mu(c_k), \int_L f d\mu + \int_L g d\mu) < \varepsilon$, for every $r_k \leq c_k, r_k \in L$, $k = 1, 2, \ldots, p$.

Now for every $r_k \leq c_k, r_k \in L$, $k = 1, 2, \ldots, p$ and $\mathcal{A}_1 \prec \mathcal{C}$, $\mathcal{A}_2 \prec \mathcal{C}$, we observe that

$$\begin{aligned} d(\sum_{k=1}^{p} (f+g)(r_k)\mu(c_k), \int_L f d\mu + \int_L g d\mu) &= d(\sum_{k=1}^{p} f(r_k)\mu(c_k) \\ &\quad + \sum_{k=1}^{p} g(r_k)\mu(c_k), \int_L f d\mu + \int_L g d\mu) \\ &\leq d(\sum_{k=1}^{p} f(r_k)\mu(c_k), \int_L f d\mu) \\ &\quad + d(\sum_{k=1}^{p} g(r_k)\mu(c_k), \int_L g d\mu) \end{aligned}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $f + g$ is μ-integrable and thus the theorem is proved.

Corollary 5.8. *If $\mu : L \to [0, \infty)$ and $f = g$, μ-almost every where f and g are two μ-integrable bounded functions on L, then*

$$\int_L f d\mu = \int_L g d\mu.$$

Proof Using Theorem 5.5, 5.6 and 5.7, we have the above corollary.

Acknowledgement

The author acknowledge with gratitude the constructive suggestions given by the reviewer towards the improvement of the manuscript.

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