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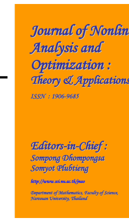

A RELATED FIXED POINT THEOREM IN n COMPLETE FUZZY METRIC SPACES

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ABSTRACT. We prove a related fixed point theorem for n mappings in n complete fuzzy metric spaces using an implicit relation which generalizes results of Aliouche and Fisher [1] and Rao et al. [13].

KEYWORDS : Fuzzy metric space; implicit relation; related fixed point
AMS Subject Classification: 47H10 54H25

1. INTRODUCTION AND PRELIMINARIES

The concept of fuzzy sets was introduced by L. Zadeh [16] in 1965. George and Veeramani [9] modified the concept of fuzzy metric spaces introduced by [11] in order to define the Hausdorff topology of fuzzy metric spaces which have very important applications in quantum particle physics particularly in connections with both string and $E-$infinity theory which were studied by El- Naschie [4, 5, 6, 7, 8] and [15]. They showed also that every metric space induces a fuzzy metric space.

Recently, Aliouche and Fisher [1], Aliouche et.al [2] and Rao et.al [13] proved some related fixed point theorems in metric spaces and fuzzy metric spaces.

Inspired by a work of Popa [12], we prove a related fixed point theorem in n complete fuzzy metric spaces using an implicit relation because it includes several contractive conditions.

Definition 1.1 ([14])**.** A binary operation $* : [0,1] \times [0,1] \longrightarrow [0,1]$ is a continuous $t-$norm if it satisfies the following conditions:

(i) $*$ is associative and commutative,
(ii) $*$ is continuous,
(iii) $a * 1 = a$ for all $a \in [0,1]$,
(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0,1]$.

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Examples of a continuous $t-$norm are $a * b = ab$ and $a * b = \min\{a, b\}$.

Definition 1.2 ([9]). The triple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary non-empty set, $*$ is a continuous $t-$norm, and M is a fuzzy set on $X^2 \times (0, \infty)$, satisfying the following conditions for each $x, y, z \in X$ and $t, s > 0$,

(FM-1) $M(x, y, t) > 0$,
(FM-2) $M(x, y, t) = 1$ if and only if $x = y$,
(FM-3) $M(x, y, t) = M(y, x, t)$,
(FM-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(FM-5) $M(x, y, .) : (0, \infty) \longrightarrow [0, 1]$ is continuous.

Note that $M(x, y, t)$ can be thought as the degree of nearness between x and y with respect to t.

Let $(X, M, *)$ be a fuzzy metric space.

1) For $t > 0$, the open ball $B(x, r, t)$ with a center $x \in X$ and a radius $0 < r < 1$ is defined by

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

2) A subset $A \subset X$ is called open if for each $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subset A$.

Let τ denote the family of all open subsets of X. Then τ is called the topology on X induced by the fuzzy metric M. This topology is Hausdorff and first countable, see [9].

Example 1.3 ([9]). Let $X = \mathbb{R}$. Denote $a * b = a.b$ for all $a, b \in [0, 1]$. Define for each $t \in (0, \infty)$ and all $x, y \in X$

$$M(x, y, t) = \frac{t}{t + |x - y|}$$

Then (X, M) is a fuzzy metric space. It is called the standard fuzzy metric induced by the metric d.

Definition 1.4 ([9]). Let $(X, M, *)$ be a fuzzy metric space.

1) A sequence $\{x_n\}$ in X converges to x if and only if for any $0 < \epsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $M(x_n, x, t) > 1 - \epsilon$; i.e., $M(x_n, x, t) \to 1$ as $n \to \infty$ for all $t > 0$.

2) A sequence $\{x_n\}$ in X is called a Cauchy sequence if and only if for any $0 < \epsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$, $M(x_n, x_m, t) > 1 - \epsilon$; i.e., $M(x_n, x_m, t) \to 1$ as $n, m \to \infty$ for all $t > 0$.

3) A fuzzy metric space (X, M, t) in which every Cauchy sequence is convergent is said to be complete.

Lemma 1.5 ([10]). *For all $x, y \in X$, $M(x, y, .)$ is a non-decreasing function.*

Definition 1.6. Let $(X, M, *)$ be a fuzzy metric space. M is said to be continuous on $X^2 \times (0, \infty)$ if

$$\lim_{n\to\infty} M(x_n, y_n, t_n) = M(x, y, t),$$

whenever $\{(x_n, y_n, t_n)\}$ is a sequence in $X^2 \times (0, \infty)$ which converges to a point $(x, y, t) \in X^2 \times (0, \infty)$; i.e.,

$$\lim_{n\to\infty} M(x_n, x, t) = \lim_{n\to\infty} M(y_n, y, t) = 1 \text{ and } \lim_{n\to\infty} M(x, y, t_n) = M(x, y, t).$$

Lemma 1.7 ([10]). *M is a continuous function on $X^2 \times (0, \infty)$.*

We denote by Ψ the set of all function $\psi : [0,1]^4 \to [0,1]$ such that
(i) ψ is upper semi continuous in each coordinate variable,
(ii) ψ is decreasing in 3rd and 4th variable,
(iii) if either $\psi(u,v,1,u) \geq 0$ or $\psi(u,1,1,v) \geq 0$ or $\psi(u,1,v,1) \geq 0$ for all $u,v \in [0,1]$, then $u \geq v$.

Example 1.8. $\psi(t_1,t_2,t_3,t_4) = t_1 - \min\{t_2,t_3,t_4\}$

Example 1.9. $\psi(t_1,t_2,t_3,t_4) = t_1 - \phi(\min\{t_2,t_3,t_4\})$,
where $\phi :]0,1] \to]0,1]$ is a increasing and continuous function with $\phi(t) > t$ for $0 < t < 1$. For example $\phi(t) = \sqrt{t}$ or $\phi(t) = t^h$ for $0 < h < 1$.

We need the following lemma of [3].

Lemma 1.10. *Let $\{x_n\}$ be a sequence in fuzzy metric space $(X,M,*)$ with $M(x,y,t) \to 1$ as $t \to \infty$ for all $x,y \in X$. If there exists a number $k \in]0,1[$ such that*

$$M(x_{n+1},x_n,kt) \geq M(x_n,x_{n-1},t).$$

Then $\{x_n\}$ is a Cauchy sequence in X.

2. MAIN RESULTS

Theorem 2.1. *Let $(X_i,M_i,\theta_i)_{1\leq i\leq n}$, be n complete fuzzy metric spaces with $M_i(x,x_i,t) \longrightarrow 1$ as $t \longrightarrow \infty$ for all $x,x_i \in X_i$ and let $\{A_i\}_{i=1}^{i=n}$ be n-mappings such that $A_i : X_i \to X_{i+1}$ for all $i = 1,..,n-1$ and $A_n : X_n \to X_1$, satisfying the inequalities*

$$\phi_1 \begin{pmatrix} M_1(A_nA_{n-1}..A_2x_2, A_nA_{n-1}..A_2A_1x_1, kt), \\ M_2(x_2, A_1A_nA_{n-1}..A_2x_2, t), \\ M_1(x_1, A_nA_{n-1}..A_2x_2, t), M_1(x_1, A_nA_{n-1}..A_2A_1x_1, t) \end{pmatrix} \geq 0 \qquad (2.1)$$

for all $x_1 \in X_1$, $x_2 \in X_2$ and $t > 0$, in general, we have

$$\phi_i \begin{pmatrix} M_i(A_{i-1}A_{i-2}..A_1A_nA_{n-1}..A_{i+1}x_{i+1}, A_{i-1}..A_1A_n..A_ix_i, kt), \\ M_{i+1}(x_{i+1}, A_iA_{i-1}..A_1A_nA_{n-1}...A_{i+1}x_{i+1}, t), \\ M_i(x_i, A_{i-1}A_{i-2}..A_1A_nA_{n-1}...A_{i+1}x_{i+1}, t), \\ M_i(x_i, A_{i-1}A_{i-2}..A_1A_nA_{n-1}...A_ix_i, t), \end{pmatrix} \geq 0 \qquad (2.i)$$

for all $x_i \in X_i$, $x_{i+1} \in X_{i+1}$, $t > 0$ and $i = 2,...,n-1$ and

$$\phi_n \begin{pmatrix} M_n(A_{n-1}A_{n-2}...A_1x_1, A_{n-1}A_{n-2}...A_1A_nx_n, kt), \\ M_1(x_1, A_nA_{n-1}A_{n-2}..A_1x_1, t), \\ M_n(x_n, A_{n-1}A_{n-2}..A_1x_1, t), \\ M_n(x_n, A_{n-1}A_{n-2}..A_1A_nx_n, t) \end{pmatrix} \geq 0 \qquad (2.n)$$

for all $x_1 \in X_1$, $x_n \in X_n$ and $t > 0$, where $\phi_i \in \Psi$, $i = 1,2,...,n$ and $0 < k < 1$. Then $A_{i-1}A_{i-2}..A_1A_nA_{n-1}..A_i$ has a unique fixed point $p_i \in X_i$ for $i = 1,...,n$. Further, $A_ip_i = p_{i+1}$ for $i = 1,...,n-1$ and $A_np_n = p_1$.

Proof. Let $\left\{x_r^{(1)}\right\}, \left\{x_r^{(2)}\right\}, ..., \left\{x_r^{(i)}\right\}, ..., \left\{x_r^{(n)}\right\}$, $r \in \mathbb{N}$ be sequences in $X_1, X_2, ..., X_i, ...X_n$ respectively. Now let $x_0^{(1)}$ be an arbitrary point in X_1, we define the sequences $\left\{x_r^{(i)}\right\}_{r\in\mathbb{N}}$ for $i = 1,...,n$ by

$$x_r^{(1)} = (A_nA_{n-1}...A_1)^r x_0^{(1)},$$

$$x_r^{(i)} = A_{i-1}A_{i-2}..A_1(A_nA_{n-1}...A_1)^r x_0^{(1)} \text{ for } i = 2,...,n.$$

For $n = 1, 2, ...,$ we assume that $x_r^{(1)} \neq x_{r+1}^{(1)}$. Applying the inequality (2.1) for $x_2 = A_1 (A_n A_{n-1}...A_1)^{r-1} x_0^{(1)}$, $x_1 = (A_n A_{n-1}...A_1)^r x_0^{(1)}$ we get

$$\begin{aligned} & \phi_1 \begin{pmatrix} M_1 \left((A_n A_{n-1}...A_1)^r x_0^{(1)}, (A_n A_{n-1}...A_1)^{r+1} x_0^{(1)}, kt \right), \\ M_2 \left(A_1 (A_n A_{n-1}...A_1)^{r-1} x_0^{(1)}, A_1 (A_n A_{n-1}...A_1)^r x_0^{(1)}, t \right), \\ M_1 \left((A_n A_{n-1}...A_1)^r x_0^{(1)}, (A_n A_{n-1}...A_1)^r x_0^{(1)}, t \right), \\ M_1 \left((A_n A_{n-1}...A_1)^r x_0^{(1)}, (A_n A_{n-1}...A_1)^{r+1} x_0^{(1)}, t \right) \end{pmatrix} \\ = & \phi_1 \left(M_1 \left(x_r^{(1)}, x_{r+1}^{(1)}, kt \right), M_2 \left(x_{r-1}^{(2)}, x_r^{(2)}, t \right), 1, M_1 \left(x_r^{(1)}, x_{r+1}^{(1)}, t \right) \right) \geq 0 \end{aligned}$$

From the implicit relation we have

$$M_1 \left(x_r^{(1)}, x_{r+1}^{(1)}, kt \right) \geq M_2 \left(x_{r-1}^{(2)}, x_r^{(2)}, t \right) \tag{3.1}$$

Applying the inequality $(2.i)$ for $x_{i+1} = A_i...A_1 (A_n...A_1)^{r-1} x_0^{(1)}$ and $x_i = A_{i-1}...A_1 (A_n...A_1)^r x_0^{(1)}$, we obtain

$$\begin{aligned} & \phi_i \begin{pmatrix} M_i \left(A_{i-1}..A_1 (A_n...A_1)^r x_0^{(1)}, A_{i-1}..A_1 (A_n...A_1)^{r+1} x_0^{(1)}, kt \right), \\ M_{i+1} \left(x_{i+1} = A_i...A_1 (A_n...A_1)^{r-1} x_0^{(1)}, A_i..A_1 (A_n...A_1)^r x_0^{(1)}, t \right), \\ M_i \left(A_{i-1}...A_1 (A_n...A_1)^r x_0^{(1)}, A_{i-1}..A_1 (A_n...A_1)^r x_0^{(1)}, t \right), \\ M_i \left(A_{i-1}...A_1 (A_n...A_1)^r x_0^{(1)}, A_{i-1}..A_1 A_{i-1}...A_1 (A_n...A_1)^{r+1} x_0^{(1)}, t \right), \end{pmatrix} \\ = & \phi_i \begin{pmatrix} M_i \left(x_r^{(i)}, x_{r+1}^{(i)}, kt \right), M_{i+1} \left(x_{r-1}^{(i+1)}, x_r^{(i+1)}, t \right), \\ 1, M_i \left(x_r^{(i)}, x_{r+1}^{(i)}, t \right) \end{pmatrix} \geq 0 \end{aligned}$$

and so

$$M_i \left(x_r^{(i)}, x_{r+1}^{(i)}, kt \right) \geq M_{i+1} \left(x_{r-1}^{(i+1)}, x_r^{(i+1)}, t \right) \tag{3.i}$$

for $i = 2, ..., n-1$ and $r = 1, 2, ...$ Now applying the inequality $(2.n)$ for $x_n = A_{n-1}...A_1 (A_n...A_1)^r x_0^{(1)}$ and $x_1 = (A_n A_{n-1}...A_1)^r x_0^{(1)}$we have

$$\begin{aligned} & \phi_n \begin{pmatrix} M_n \left(A_{n-1}...A_1 (A_n...A_1)^r x_0^{(1)}, A_{n-1}...A_1 (A_n...A_1)^{r+1} x_0^{(1)}, kt \right), \\ M_1 \left((A_n...A_1)^r x_0^{(1)}, (A_n...A_1)^{r+1} x_0^{(1)}, t \right), \\ M_n \left(x_n = A_{n-1}...A_1 (A_n...A_1)^r x_0^{(1)}, A_{n-1}..A_1 (A_n...A_1)^r x_0^{(1)}, t \right), \\ M_n \left(A_{n-1}...A_1 (A_n...A_1)^r x_0^{(1)}, A_{n-1} A_{n-2}..A_1 (A_n A_{n-1}...A_1)^{r+1} x_0^{(1)}, t \right) \end{pmatrix} \\ = & \phi_n \begin{pmatrix} M_n \left(x_r^{(n)}, x_{r+1}^{(n)}, kt \right), M_1 \left(x_{r-1}^{(1)}, x_r^{(1)}, t \right), \\ 1, M_n \left(x_r^{(n)}, x_{r+1}^{(n)}, t \right) \end{pmatrix} \geq 0 \end{aligned}$$

and so

$$M_n \left(x_r^{(n)}, x_{r+1}^{(n)}, kt \right) \geq M_1 \left(x_{r-1}^{(1)}, x_r^{(1)}, t \right) \tag{3.n}$$

It now follows from $(3.1), (3.i)$ and $(3.n)$ that for large enough n we obtain

$$\begin{aligned} M_1 \left(x_r^{(1)}, x_{r+1}^{(1)}, kt \right) & \geq M_2 \left(x_{r-1}^{(2)}, x_r^{(2)}, t \right) \\ M_i \left(x_r^{(i)}, x_{r+1}^{(i)}, t \right) & \geq M_{i+1} \left(x_{r-1}^{(i+1)}, x_r^{(i+1)}, \frac{t}{k} \right) \\ & \geq ... \end{aligned}$$

$$
\begin{aligned}
&\geq M_n\left(x_{r+i-n}^{(n)}, x_{r+i-n+1}^{(n)}, \frac{t}{k^{n-i}}\right) \\
&\geq M_1\left(x_{r+i-n-1}^{(1)}, x_{r+i-n}^{(1)}, \frac{t}{k^{n-i+1}}\right) \\
&\geq \ldots.. \\
&\quad M_1\left(x_{r+i-2n-1}^{(1)}, x_{r+i-2n}^{(1)}, \frac{t}{k^{2n-i+1}}\right) \\
&\geq \ldots. \\
&\geq M_1\left(x_{r+i-mn-1}^{(1)}, x_{r+i-mn}^{(1)}, \frac{t}{k^{mn-i+1}}\right) \\
&\geq \min\left\{M_1\left(x_1^{(1)}, x_2^{(1)}, \frac{t}{k^{mn}}\right), \ldots., M_n\left(x_1^{(n)}, x_2^{(n)}, \frac{t}{k^{mn}}\right)\right\}
\end{aligned}
$$

Since $0 < k < 1$. it follows from lemma 1.10 that $\left\{x_r^{(i)}\right\}$ is a Cauchy sequences in X_i with a limit p_i in X_i for $i = 1, 2, .., n$.

To prove that p_i is a fixed point of $A_{i-1}...A_1A_n...A_ip_i$ for $i = 2, ..., n-1$, suppose that $A_{i-1}...A_1A_n...A_ip_i \neq p_i$. Using the inequality $(2.i)$ for $x_i = p_i$ and $x_{i+1} = x_r^{(i+1)}$we obtain

$$
\phi_i\left(\begin{array}{c} M_i\left(x_r^{(i)}, A_{i-1}..A_1A_n...A_ip_i, kt\right) \\ , M_{i+1}\left(x_r^{(i+1)}, x_{r+1}^{(i+1)}, t\right), M_i\left(p_i, x_r^{(i)}, t\right), \\ M_i\left(p_i, A_{i-1}..A_1A_n...A_ip_i, t\right) \end{array}\right) \geq 0
$$

Letting $r \to \infty$ we have

$$
\phi_i\left(\begin{array}{c} M_i\left(p_i, A_{i-1}A_{i-2}..A_1A_nA_{n-1}...A_ip_i, kt\right), 1, 1, \\ d_i\left(p_i, A_{i-1}A_{i-2}..A_1A_nA_{n-1}...A_ip_i, t\right) \end{array}\right) \geq 0.
$$

It follows from (iii) that $p_i = A_{i-1}A_{i-2}..A_1A_nA_{n-1}...A_ip_i$ in X_i for $i = 2, ..., n-1$ and $p_i = A_{i-1}p_{i-1} = ... = A_{i-1}A_{i-2}....A_2A_1p_1$.

For the case $i = 1$, we use (2.1) for $x_1 = p_1$ and $x_2 = A_1\left(A_nA_{n-1}...A_1\right)^{r-1}x_0^{(1)} = x_r^{(2)}$giving

$$
\phi_1\left(\begin{array}{c} M_1\left(x_r^{(1)}, A_nA_{n-1}...A_1p_1, kt\right), \\ M_2\left(x_r^{(2)}, x_{r+1}^{(2)}, t\right), M_1\left(p_1, x_r^{(1)}, t\right), \\ M_1\left(p_1, A_nA_{n-1}...A_1p_1, t\right) \end{array}\right) \geq 0
$$

letting $r \to \infty$ we have

$$
\phi_1\left(\begin{array}{c} M_1\left(p_1, A_nA_{n-1}...A_1p_1, kt\right), 1, 1, \\ M_1\left(p_1, A_nA_{n-1}...A_1p_1, t\right) \end{array}\right) \geq 0.
$$

It follows from (iii) that $A_nA_{n-1}...A_2A_1p_1 = p_1$ in X_1.

Finally, if $i = n$, using the inequality $(2.n)$ for $x_n = p_n$ and $x_1 = x_r^{(1)}$ we get

$$
\phi_n\left(\begin{array}{c} M_n\left(x_{r+1}^{(n)}, A_{n-1}A_{n-2}..A_1A_np_n, kt\right), \\ M_1\left(x_r^{(1)}, x_{r+1}^{(1)}, t\right), M_n\left(p_n, x_{r+1}^{(n)}, t\right), \\ M_n\left(p_n, A_{n-1}A_{n-2}..A_1A_np_n, t\right) \end{array}\right) \geq 0.
$$

Letting $r \to \infty$ we have

$$
\phi_n\left(\begin{array}{c} M_n\left(p_n, A_{n-1}A_{n-2}..A_1A_np_n, kt\right), 1, 1, \\ M_n\left(p_n, A_{n-1}A_{n-2}..A_1A_np_n, t\right) \end{array}\right) \geq 0.
$$

and by (iii), $p_n = A_{n-1}A_{n-2}..A_1A_np_n$ in X_n and $p_n = A_{n-1}p_{n-1} = ... = A_{n-1}A_{n-2}....A_2A_1p_1$.

To prove the uniqueness, suppose that $A_{i-1}....A_1A_n...A_i$ has a second fixed point $z_1 \neq p_1$ in X_1. Using the inequality (2.1) for $x_{i+1} = A_iz_i$ and $x_i = p_i$ we get

$$\phi_i \begin{pmatrix} M_i\left(A_{i-1}....A_1A_n...A_iz_i, A_{i-1}....A_1A_n...A_ip_i, kt\right), \\ M_{i+1}\left(A_iz_i, A_{i-1}....A_1A_n...A_iz_i, t\right) \\ M_i\left(p_i, A_{i-1}....A_1A_n...A_iz_i, t\right), M_1\left(p_i, A_{i-1}....A_1A_n...A_ip_i, t\right) \end{pmatrix} \geq 0$$

and so

$$\phi_i\left(M_i\left(z_i, p_i, kt\right), 1, M_i\left(p_i, z_i, t\right), 1\right) \geq 0$$

which implies that $z_i = p_i$, proving the uniqueness of p_i in X_i for $i = 2, ..., n-1$. The uniqueness of p_1 in X_1 and p_n in X_n follow similarly.

Finally, we note that

$$A_ip_i = A_iA_{i-1}...A_1A_n...A_{i+1}\left(A_ip_i\right),$$

hence, p_i is a fixed point of $A_i...A_1A_n...A_{i+1}$. Since the fixed point is unique, it follows that $A_ip_i = p_{i+1}$ for all $i = 1, ..., n-1$. It follows similarly that $A_np_n = p_1$. This complete the proof of the theorem. □

Example 2.2. Let (M_i, X_i, θ_i) for $i = 1, ..., n$ be n fuzzy metric spaces, where $M_i\left(x_i, y_i, t\right) = \dfrac{t}{t + |x_i - y_i|}$ and $X_i = \{x_i : i-1 \leq x_i \leq i\}$ for $i = 1, ..., n$. Define $A_i : X_i \to X_{i+1}$ for $i = 1, ..., n-1$ and $A_n : X_n \to X_1$ by

$$A_1x_1 = \begin{cases} \dfrac{5}{4} & \text{if } 0 \leq x_1 < \dfrac{1}{2}, \\ \dfrac{3}{2} & \text{if } \dfrac{1}{2} \leq x_1 \leq 1 \end{cases},$$

$$A_ix_i = \begin{cases} i + \dfrac{1}{4} & \text{if } i-1 \leq x_i < i - \dfrac{3}{4}, \\ i + \dfrac{1}{2} & \text{if } i - \dfrac{3}{4} \leq x_i \leq i \end{cases} \quad \text{for all } i = 2, ..., n-1,$$

$$A_nx_n = \begin{cases} \dfrac{3}{4} & \text{if } n-1 \leq x_n < n - \dfrac{3}{4}, \\ 1 & \text{if } n - \dfrac{3}{4} \leq x_n \leq n \end{cases}$$

Let $\phi_1 = \phi_2 = ... = \phi_n = \phi$ and $\phi\left(t_1, t_2, t_3, t_4\right) = t_1 - \min\{t_2, t_3, t_4\}$. Note that there exists p_i in X_i such that $\left(A_{i-1}A_{i-2}..A_1A_n...A_i\right)p_i = p_i$ for $i = 1, ..., n$. For example If we put $i = n$, we get $\left(A_{n-1}A_{n-2}..A_1A_n\right)p_n = p_n$ if $p_n = n - \dfrac{1}{2}$ because

$$\begin{aligned} A_{n-1}A_{n-2}..A_1A_n\left(n - \frac{1}{2}\right) &= A_{n-1}A_{n-2}..A_1\left(1\right), \\ &= A_{n-1}A_{n-2}..A_2\left(\frac{3}{2}\right) \\ &\vdots \\ &= A_{n-1}A_{n-2}..A_{i+1}\left(i + \frac{1}{2}\right) \\ &\vdots \end{aligned}$$

$$\begin{aligned} &= A_{n-1}A_{n-2}\left(n-\frac{5}{2}\right) \\ &= A_{n-1}\left(n-\frac{3}{2}\right) \\ &= n-\frac{1}{2} \quad \text{car } n-\frac{7}{4}\leq n-\frac{3}{2}\leq n-1. \end{aligned}$$

Note that for all $i=1,....,n-1$ and $i-\frac{3}{4}\leq x_i<i$; $(i+1)-\frac{3}{4}\leq A_ix_i<i+1$ and $\frac{1}{2}\leq A_nx_n\leq 1$ with $n-\frac{3}{4}\leq x_n<n$, there exists $p_i=i-\frac{1}{2}$ such that $(A_{i-1}..A_1A_n...A_i)\left(i-\frac{1}{2}\right)=i-\frac{1}{2}$ for $i=1,...,n-1$.

The inequalities $(1.i)$ for all $i=1,...,n$ are satisfied since the value of the left hand side of each inequality is 1. In fact $M_i\left(A_{i-1}...A_1A_n..A_{i+1}x_{i+1},A_{i-1}...A_1A_n..A_ix_i,t\right)=1$ for $i=1,....,n-1$ because

(1) If $i-1\leq x_i<i-\frac{3}{4}$ we have

$$\begin{aligned} A_{i-1}...A_1A_n...A_ix_i &= A_{i-1}...A_1A_nA_{n-1}...A_{i+1}\left(i+\frac{1}{4}\right) \\ &= A_{i-1}A_{i-2}...A_1A_nA_{n-1}..A_{i+2}\left((i+1)+\frac{1}{2}\right) \\ &\vdots \\ &= A_{i-1}A_{i-2}...A_1A_nA_{n-1}\left((n-2)+\frac{1}{2}\right) \\ &= A_{i-1}A_{i-2}...A_1A_n\left(n-\frac{1}{2}\right) \\ &= A_{i-1}A_{i-2}...A_2A_1\left(1\right), \\ &= A_{i-1}A_{i-2}...A_2\left(1+\frac{1}{2}\right) \\ &\vdots \\ &= A_{i-1}\left((i-2)+\frac{1}{2}\right) \\ &= (i-1)+\frac{1}{2}=i-\frac{1}{2}. \end{aligned}$$

(2) If $i-\frac{3}{4}\leq x_i\leq i$, we get

$$\begin{aligned} A_{i-1}A_{i-2}...A_1A_nA_{n-1}..A_ix_i &= A_{i-1}A_{i-2}...A_1A_nA_{n-1}..A_{i+1}\left(i+\frac{1}{2}\right) \\ &\vdots \\ &= A_{i-1}A_{i-2}...A_1A_nA_{n-1}\left((n-2)+\frac{1}{2}\right) \\ &= A_{i-1}A_{i-2}...A_1A_n\left(n-\frac{1}{2}\right), \\ &= A_{i-1}A_{i-2}...A_2A_1\left(1\right)= \end{aligned}$$

$$\vdots$$
$$A_{i-1}\left((i-2)+\frac{1}{2}\right) = (i-1)+\frac{1}{2} = i-\frac{1}{2}.$$

(3) if $i \leq x_{i+1} < i+\frac{1}{4}$ we get

$$\begin{aligned}
A_{i-1}...A_1A_n..A_{i+1}x_{i+1} &= A_{i-1}...A_1A_n..A_{i+2}\left((i+1)+\frac{1}{4}\right)\\
&\vdots\\
&= A_{i-1}...A_1A_n\left(n-\frac{1}{2}\right)\\
&= A_{i-1}...A_2A_1(1) =\\
&\vdots\\
&= A_{i-1}\left((i-2)+\frac{1}{2}\right)\\
&= (i-1)+\frac{1}{2} = i-\frac{1}{2}.
\end{aligned}$$

(4) If $i+\frac{1}{4} \leq x_{i+1} \leq i+1$ we obtain

$$\begin{aligned}
A_{i-1}...A_1A_n..A_{i+1}x_{i+1} &= A_{i-1}...A_1A_n..A_{i+2}\left((i+1)+\frac{1}{2}\right)\\
&\vdots\\
&= A_{i-1}...A_1A_n\left(n-\frac{1}{2}\right),\\
&= A_{i-1}...A_2A_1(1) =\\
&\vdots\\
&= A_{i-1}\left((i-2)+\frac{1}{2}\right)\\
&= (i-1)+\frac{1}{2} = i-\frac{1}{2}.
\end{aligned}$$

Thus, all the conditions of theorem 2.1 are satisfied.

If we take $n=5$ in theorem 2.1, we get the following corollary.

Corollary 2.3. *Let* (X_i, M_i, θ_i), $i=1,...,5$ *be* 5 *complete fuzzy metric spaces,* $A_i : X_i \to X_{i+1}$, $i=1,2,3,4$ *and* $A_5 : X_5 \to X_1$ *be* 5 *mappings satisfying*

$$\phi_1\begin{pmatrix} M_1(A_5A_4A_3A_2x_2, A_5A_4A_3A_2A_1x_1, kt),\\ M_1(x_1, A_5A_4A_3A_2x_2, t),\\ M_1(x_1, A_5A_4A_3A_2A_1x_1, t),\\ M_2(x_2, A_1A_5A_4A_3A_2x_2, t)\end{pmatrix} \geq 0 \tag{4.1}$$

for all $x_1 \in X_1$ *and* $x_2 \in X_2$,

$$\phi_2\begin{pmatrix} M_2(A_1A_5A_4A_3x_3, A_1A_5A_4A_3A_2x_2, kt),\\ M_3(x_3, A_2A_1A_5A_4A_3x_3, t),\\ M_2(x_2, A_1A_5A_4A_3A_2x_2, t)\\ M_2(x_2, A_1A_5A_4A_3x_3, t)\end{pmatrix} \geq 0 \tag{4.2}$$

for all $x_2 \in X_2$ *and* $x_3 \in X_3$

$$\phi_3 \begin{pmatrix} M_3\left(A_2A_1A_5A_4x_4, A_2A_1A_5A_4A_3x_3, kt\right), \\ M_4\left(x_4, A_3A_2A_1A_5A_4x_4, t\right), \\ M_3\left(x_3, A_2A_1A_5A_4A_3x_3, t\right), \\ M_3\left(x_3, A_2A_1A_5A_4x_4, t\right) \end{pmatrix} \geq 0 \tag{4.3}$$

for all $x_3 \in X_3$ *and* $x_4 \in X_4$

$$\phi_4 \begin{pmatrix} M_4\left(A_3A_2A_1A_5x_5, A_3A_2A_1A_5A_4x_4, kt\right), \\ M_5\left(x_5, A_4A_3A_2A_1A_5x_5, t\right), \\ M_4\left(x_4, A_3A_2A_1A_5A_4x_4, t\right), \\ M_4\left(x_4, A_3A_2A_1A_5x_5, t\right) \end{pmatrix} \geq 0 \tag{4.4}$$

for all $x_4 \in X_4$ *and* $x_5 \in X_5$

$$\phi_5 \begin{pmatrix} M_5\left(A_4A_3A_2A_1x_1, A_4A_3A_2A_1A_5x_5, kt\right), \\ M_1\left(x_1, A_5A_4A_3A_2A_1x_1, t\right), \\ M_5\left(x_5, A_4A_3A_2A_1A_5x_5, t\right), \\ M_5\left(x_5, A_4A_3A_2A_1x_1, t\right) \end{pmatrix} \geq 0 \tag{4.5}$$

for all $x_1 \in X_1$, $x_5 \in X_5$ *and for all* $t > 0$, *where* $0 < k < 1$. *Then*

(a_1) $A_5A_4A_3A_2A_1$ *has a unique fixed point* $w_1 \in X_1$,
(a_2) $A_1A_5A_4A_3A_2$ *has a unique fixed point* $w_2 \in X_2$,
(a_3) $A_2A_1A_5A_4A_3$ *has a unique fixed point* $w_3 \in X_3$,
(a_4) $A_3A_2A_1A_5A_4$ *has a unique fixed point* $w_4 \in X_4$,
(a_5) $A_4A_3A_2A_1A_5$ *has a unique fixed point* $w_5 \in X_5$,
Further, $A_1w_1 = w_2$, $A_2w_2 = w_3$, $A_3w_3 = w_4$, $A_4w_4 = w_5$ *and* $A_5w_5 = w_1$.

If we take $n = 2$ in theorem 2.1, we obtain theorem 2.9 of Rao et al. [13] and a fuzzy version of theorem 3 of [1].

The following example illustrates our corollary 2.3.

Example 2.4. Let (M_i, X_i, θ_i) for $i = 1, ..., 5$ be 5 fuzzy metric spaces where $M_i\left(x_i, y_i, t\right) = \dfrac{t}{t + |x_i - y_i|}$ and $X_i = \{x_i : i - 1 \leq x_i \leq i\}$ for $i = 1, ..., 5$. Define $A_i : X_i \to X_{i+1}$ for $i = 1, ..., 4$ and $A_5 : X_5 \to X_1$ by

$$A_1x_1 = \begin{cases} 1 \text{ if } x_1 \in [0, \frac{3}{4}[\\ \\ \frac{3}{2} \text{ if } x_1 \in [\frac{3}{4}, 1] \end{cases}, \ A_2x_2 = \begin{cases} \frac{5}{2} \text{ if } x_2 \in [1, \frac{3}{2}[\\ \\ 3 \text{ if } x_2 \in [\frac{3}{2}, 2] \end{cases}$$

$$A_3x_3 = \begin{cases} \frac{13}{4} \text{ if } x_3 \in [2, \frac{5}{2}[\\ \\ \frac{7}{2} \text{ if } x_3 \in [\frac{5}{2}, 3] \end{cases}, \ A_4x_4 = \begin{cases} \frac{17}{4} \text{ if } x_4 \in [3, \frac{7}{2}[\\ \\ \frac{9}{2} \text{ if } x_4 \in [\frac{7}{2}, 4] \end{cases}$$

$$A_5x_5 = \begin{cases} \frac{3}{4} \text{ if } x_5 \in [4, \frac{9}{2}[\\ \\ 1 \text{ if } x_5 \in [\frac{9}{2}, 5] \end{cases}$$

Let $\phi_1\left(t_1, t_2, t_3, t_4\right) = t_1 - \min\left\{t_2, t_3, t_4\right\}$ and $\phi_1 = \phi_2 = \phi_3 = \phi_4 = \phi_5$

Further, the inequalities $(4.1), (4.2), (4.3), (4.4), (4.5)$ are satisfied since the left hand side of each inequality is 1 and

$$A_5A_4A_3A_2A_1\left(1\right) = 1$$

$$
\begin{aligned}
A_1A_5A_4A_3A_2\left(\frac{3}{2}\right) &= \frac{3}{2}\\
A_2A_1A_5A_4A_3\left(\frac{5}{2}\right) &= \frac{5}{2}\\
A_3A_2A_1A_5A_4\left(\frac{7}{2}\right) &= \frac{7}{2}\\
A_4A_3A_2A_1A_5\left(\frac{9}{2}\right) &= \frac{9}{2}.
\end{aligned}
$$

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