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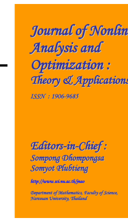

STOCHASTIC MODEL FOR GOLD PRICES AND ITS APPLICATION FOR NO-ARBITRAGE GOLD DERIVATIVE PRICING

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ABSTRACT. In this paper, we develop a one-factor model of stochastic behavior of gold prices. The gold prices are assumed to follow an extended Geometric Brownian Motion with a time-varying drift which describes seasonal variation in gold prices. The drift includes instantaneous convenience yields which follow an ordinary differential equation. Moreover, we derive closed-form solutions for no-arbitrage prices of gold futures and European gold options under the no-arbitrage assumptions.

KEYWORDS : Gold futures; European options; No-arbitrage prices; Instantaneous convenience yields.

1. INTRODUCTION

Gold has been a valuable metal throughout the ages because of its versatility. It can be used in many applications. One of its primary uses is as jewelry and adornment. It is even used in aerospace, dentistry, electronics. People can also use gold as the standard value for the money of each country. Furthermore, gold is extremely important to the development of industry and technology of each country because it is considered as a liquid asset and has low fluctuations. Therefore it is used as an assurance when issuing bank notes. We can see that in many countries they have gold for a guaranteed source of investment funds. Moreover, gold investment is a good way for getting high return in the long run. Nowadays,

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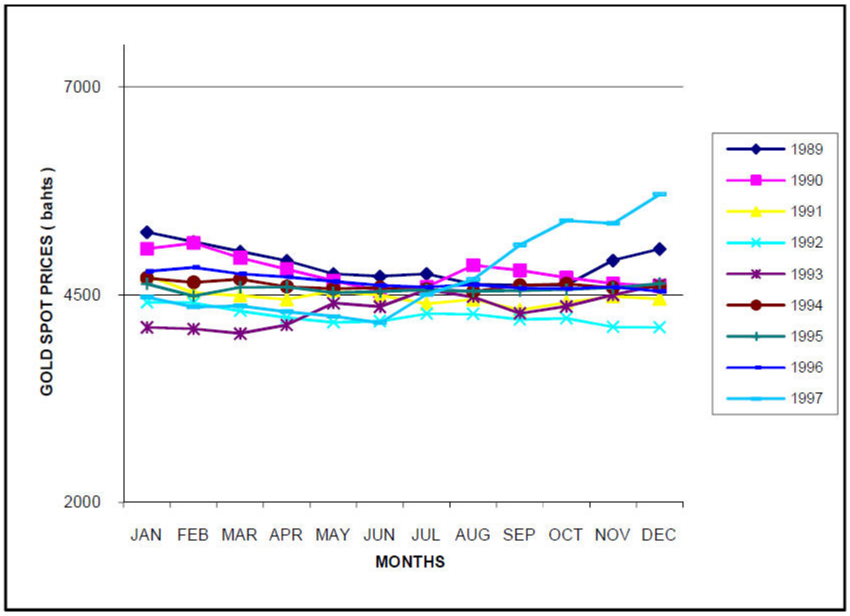

FIGURE 1. Monthly averages of gold spot prices over nine years

the investment in commodity derivatives markets which have gold as an underlying asset is receiving widespread attention. Gold derivatives markets have witnessed a tremendous growth in recent years. In order to provide closed-form solutions for gold derivatives such as futures and options, one can treat gold spot prices as a random walk. In other words, gold spot prices are assumed to follow a stochastic differential equation (SDE). Using the SDE theory, the closed-form solutions can be obtained by solving the associated partial differential equations.

By considering gold as one of those commodities, one can choose a model proposed by Brennan-Schwartz (1985) [1] to describe the dynamics of gold spot prices To follow the model, gold spot prices are assumed to follow a Geometric Brownian Motion (GBM) and the convenience yield, which is an important factor influencing commodity prices, is described in the same way as a dividend yield. Nevertheless, this specification is inappropriate because it does not take into account the mean-reversion property of commodity prices. Schwartz (1997) [2] introduced variation of this model in which the convenience yield is mean reverting and intervenes in the commodity price dynamics. Besides the mean reversion property of commodity prices, the other main empirical characteristic that makes commodities noticeable different from stocks, bonds and other financial assets, is seasonality in prices. Many commodities, such as agricultural commodities or natural gas, exhibit seasonality in prices, due to harvest cycles in the case of agricultural commodities and change consumptions in the case of natural gas. In addition, gold also have seasonal variation in prices as well (see Figure 1.). We use the gold spot prices data [6] during January 1989 to December 1997 in plotting graph to observe the tendency of gold spot prices shown in Figure 1.. It shows seasonal variation of gold prices. From analyzing the graph, we can see that gold spot prices at the beginning of each year are high. They drop in the middle and at the end of each years, they are high again. Thus, the graph looks like sine wave. Therefore, a model of gold spot prices introduced in this paper will take to account a seasonal variation in gold prices.

The remaining of this paper is organizes as follows. In section 2, we present a one-factor model, which is an extension of Schwartz model 1 [2]. In section 3, we present the no-arbitrage assumptions and derive for closed-form solutions for no-arbitrage prices of gold futures and European gold options. Finally, We conclude the paper in section 4.

2. STOCHASTIC MODEL FOR GOLD PRICES

In this section, we develop a one-factor model of stochastic behavior gold prices. The gold prices are assumed to follow an extended Geometric Brownian Motion with a time-varying drift which describes seasonal variation in gold prices. The drift includes instantaneous convenience yields which follow an ordinary differential equation (ODE). Our model can be written as follow: under an equivalent martingale measure $\mathbb{Q}$,

$$dS_t = (r - \delta(t))S_t dt + \sigma S_t dW_t, S_0 = s_0 > 0, \tag{2.1}$$

$$\frac{d\delta(t)}{dt} = \kappa\left(\alpha(t) - \delta(t)\right), \delta\left(0\right) = \delta_0, \tag{2.2}$$

$$\alpha\left(t\right) = \alpha_0 + \alpha_1 sin(2\pi(t - t_\alpha)), \tag{2.3}$$

where $(S_t(\omega))_{t\in[0,T]}$ is a gold spot price process, r is a risk-free interest rate, $\delta(t)$ is instantaneous convenience yield at time T which is assumed to follow the ordinary differential equation (2) and has mean reversion property, $(W_t(\omega))_{t\in[0,T]}$ is a-one dimensional standard Brownian motion under a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ and a filtration $(\mathcal{F})_{t\geq 0}$, κ is the speed of adjustment of the gold prices, α_0 is a long run mean, σ is the volatility of gold prices, $\alpha(t)$ is seasonal variation in in convenience yields. The parameters α_1 and T denoted, respectively, the annual seasonality parameters and the seasonality centering parameter, representing the time of annual peak of equilibrium price in a year.

From (2), the closed-form solution of the ODE (2) is

$$\delta(t) = \alpha_0 + C(t) + (\delta_0 - \alpha_0 - C(0))e^{-\kappa t}, \tag{2.4}$$

where

$$C(t) = \frac{-2\alpha_1\kappa\pi cos(2\pi(t - t_\alpha)) + \alpha_1\kappa^2 sin(2\pi(t - t_\alpha))}{\kappa^2 + 4\pi^2}. \tag{2.5}$$

From Kloeden and Platen [4 page 120], the strong solution of the SDE (1) can be expressed as

$$S_t = S_{t_0} e^{\int_{t_0}^{t}(r - \frac{\sigma^2}{2} - \delta(s))ds + \sigma\sqrt{(t - t_0)}Z}, \tag{2.6}$$

for all $0 \leq t_0 \leq t \leq T$, where Z is the standard normal random variable.

Proposition 2.1. *The gold spot price S_t modeled by (1) is neither negative nor zero for all $t \geq 0$. Moreover, for a fixed $t_0 \geq 0, ln(S_t/S_{t_0})$ is normal distributed with mean $\int_{t_0}^{t}(r - \frac{\sigma^2}{2} - \delta(s))ds$ and variance $\sigma^2(t - t_0)$.*

3. VALUATION OF GOLD DERIVATIVES

In order to price futures and options of the commodity (gold in this case), we assume the following assumptions hold.

The no-arbitrage assumptions

1. The market is arbitrage-free, that is for any portfolio $\varphi = (\varphi_i), V_\varphi(0) = 0$

and $V_\varphi(T) \geq 0, \mathbb{P}-$a.s. for all time $T > 0$ imply $V_\varphi(T) = 0, \mathbb{P}-$ a.s., where $V_\varphi(t) \equiv V_\varphi(t, S_t, \varphi_t)$ denotes the value of the portfolio φ at time t and $\mathbb{P}$ denotes an original probability measures. Namely, if a portfolio requires a null investment and is riskless (there is no possible loss at the time horizon T), then its terminal value at time T has to be zero.
2. The market participants are subject to no transaction costs when they trade.
3. The market participants are subject to the same or no tax rate on all net trading profits.
4. The market participants can borrow/lend money at the same risk-free rate of interest.

A. GOLD FUTURES PRICING

Form Rujivan [5], under the no-arbitrage assumptions in a futures market, the no-arbitrage price (fair-price) of gold futures at a current time t, denoted by $F^T(t, S_t)$, satisfies,

$$F^T(t, S_t) = E_{\mathbb{Q}}[S_T|\mathcal{F}_t], \tag{3.1}$$

where the expectation is taken under the equivalent martingale measure $\mathbb{Q}$ conditioned on $\mathcal{F}_t$.

The relation (7) implies that F^T solves the partial differential equation:

$$-\frac{\partial F^T}{\partial t} = (r - \delta(t))S\frac{\partial F^T}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial F^T}{\partial S^2} \tag{3.2}$$

subject to a terminal condition

$$F^T(T, S) = S, \tag{3.3}$$

for all $S \geq 0$.

We now solve the PDE (8) subject to (9) to obtained closed-form solutions for no-arbitrage gold futures price.

Theorem 3.1. *(Determination of gold futures prices)*
For given and fixed maturity date T, no-arbitrage gold futures prices to the PDE (8) can be expressed as

$$F^T(t, S_t) = S_t e^{A(T-t)} \tag{3.4}$$

where

$$A(\tau) = r\tau - \int_0^\tau \delta(T - s)ds, \tag{3.5}$$

for all $\tau \geq 0$.

Proof. To avoid confusion about the notations, we omit writing the subscript t of S_t and write $F^T = F^T(t, S_t)$.
Let $\tau = T - t$, and we calculate

$$\frac{\partial F^T}{\partial t} = -(F^T A'(\tau)), \quad \frac{\partial F^T}{\partial S} = \frac{\partial F^T}{S}, \quad \frac{\partial^2 F^T}{\partial S^2} = 0,$$

where $' = \frac{d}{d\tau}$.
Replacing the above partial derivatives into (8), we then obtain the following ODE,

$$A'(\tau) = (r - \delta(T - \tau)) \tag{3.6}$$

and the terminal condition implies that

$$A(0) = 0. \tag{3.7}$$

Then, we solve (12) subject to (13) to obtain (11). □

B. EUROPEAN GOLD OPTIONS PRICING

In this section, we consider European options written on gold spot prices. We first consider a European call option. Let t and K be respectively, a maturity date and strike price of the call option. From [5], under the no-arbitrage assumptions, the call option price must equal to the present value of the expected payoff of the call option under the equivalent martingale measure $\mathbb{Q}$,

$$C(T, t, S_t, K) = e^{-r(T-t)} E_{\mathbb{Q}}[max(0, S_T - K)|\mathcal{F}_t], \tag{3.8}$$

Theorem 3.2. *(No-arbitrage prices of European call options for gold) The no-arbitrage European gold call options prices with strike price K is*

$$C(T, t, S_t, K) = S_t e^{-\int_t^T \delta(s)ds} \Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) \tag{3.9}$$

where

$$d_1 = \frac{ln(\frac{S_t}{K}) + r(T-t) - \int_t^T \delta(s)ds + \frac{\sigma^2(T-t)}{2}}{\sigma\sqrt{T-t}} \tag{3.10}$$

$$d_2 = d_1 - \sigma\sqrt{T-t} \tag{3.11}$$

Proof. We omit writing the martingale measure $\mathbb{Q}$ and the filtration $\mathcal{F}_t$ in this proof to avoid confusion about notations. From (14), we obtain

$$\begin{aligned} C(T, t, S_t, K) &= e^{-r(T-t)} E[max(0, S_T - K)], \\ &= e^{-r(T-t)} E[(S_T - K)^+], \\ &= e^{-r(T-t)} E[I(S_T - K)], \\ &= e^{-r(T-t)} I[S_T] - Ke^{-r(T-t)} E[I], \end{aligned} \tag{3.12}$$

where I is the indicator random variable for the event that the option finishes in the money, that is,

$$I = \begin{cases} 1, & \text{if} \quad S_T > K, \\ 0, & \text{if} \quad S_T \leq K. \end{cases} \tag{3.13}$$

Next, we will show that

$$I = \begin{cases} 1, & \text{if} \quad Z > \sigma\sqrt{T-t} - d_1, \\ 0, & \text{if} \quad otherwise. \end{cases} \tag{3.14}$$

where d_1 is given in (16).
Using (6) and (19) , we obtain

$$\begin{aligned} S_T > K &\Leftrightarrow S_t e^{\int_t^T (r - \frac{\sigma^2}{2} - \delta(s))ds + \sigma\sqrt{T-t}Z} > K \\ &\Leftrightarrow \int_t^T (r - \frac{\sigma^2}{2} - \delta(s))ds + \sigma\sqrt{T-t}Z > ln(\frac{K}{S_t}) \\ &\Leftrightarrow Z > \frac{ln(\frac{K}{S_t}) - r(T-t) + \int_t^T \delta(s)ds + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \\ &\Leftrightarrow Z > \sigma\sqrt{T-t} - d_1 \end{aligned}$$

and we also obtain (20). From (20), we have

$$E[I] = P(S_T > K)$$

$$\begin{aligned} &= P(Z > \sigma\sqrt{T-t} - d_1) \\ &= P(Z < d_1 - \sigma\sqrt{T-t}) \\ &= \Phi(d_1 - \sigma\sqrt{T-t}) \\ &= \Phi(d_2), \end{aligned} \tag{3.15}$$

where Φ is the standard normal distribution function.
Using (6) and (20) with $a = \sigma\sqrt{T-t} - d_1$,
we then obtain,

$$\begin{aligned} e^{-r(T-t)}E[IS_T] &= e^{r(T-t)} \int_a^\infty S_t e^{r(T-t) - \int_t^T \delta(s)ds + \frac{1}{2}\sigma^2(T-t) + \sigma\sqrt{T-t}y} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\ &= S_t e^{-\int_t^T \delta(s)ds} \frac{1}{\sqrt{2\pi}} \int_a^\infty e^{-\frac{1}{2}(y^2 - 2\sigma\sqrt{T-t}y + \sigma^2(T-t))} dy \\ &= S_t e^{-\int_t^T \delta(s)ds} \frac{1}{\sqrt{2\pi}} \int_a^\infty e^{\frac{-(y-\sigma\sqrt{T-t})^2}{2}} dy \\ &= S_t e^{-\int_t^T \delta(s)ds} \frac{1}{\sqrt{2\pi}} \int_{-d_1}^\infty e^{\frac{-x^2}{2}} dx \\ &= S_t e^{-\int_t^T \delta(s)ds} P(Z > -d_1) \\ &= S_t e^{-\int_t^T \delta(s)ds} P(Z < d_1) \\ &= S_t e^{-\int_t^T \delta(s)ds} \Phi(d_1) \end{aligned} \tag{3.16}$$

Replacing (21) and (22) into (18), we then reach (15). □

Theorem 3.3. *(No-arbitrage prices of European put options for gold) The no-arbitrage European gold put options prices with strike price K is*

$$P(T, t, S_t, K) = Ke^{-r(T-t)}\Phi(-d_2) - S_t(1 - e^{-\int_t^T \delta(s)ds}\Phi(d_1)), \tag{3.17}$$

where d_1 and d_2 satisfy, respectively, (16) and (17).

Proof. The put-call parity [3] equation satisfies

$$C(T, t, S_t, K) + Ke^{-r(T-t)} = P(T, t, S_t, K) + S_t. \tag{3.18}$$

Replacing (15) into (24) , we then obtain

$$P(T, t, S_t, K) = (Ke^{-r(T-t)} - Ke^{-r(T-t)}\Phi(d_2)) - (S_t - S_t e^{-\int_t^T \delta(s)ds}\Phi(d_1)).$$

Finally, we rearrange the equation above to obtain the no- arbitrage European gold put options prices. □

4. CONCLUSION

This paper has developed a one-factor model for gold prices which is an extension of the model proposed by Schwartz. We have provided closed-form solutions for no-arbitrage prices of gold futures and European gold options under the no-arbitrage assumptions. Moreover, one can use our model to predict gold prices in the futures if the unknown parameters are estimated using historical data of gold spot prices. Finally, closed-form no-arbitrage prices of American options for gold will be derived in our future work

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