
**SOME DOUBLE SEQUENCE SPACES IN n -NORMED SPACES USING IDEAL
CONVERGENCE AND A SEQUENCE OF ORLICZ FUNCTIONS**

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ABSTRACT. In the present paper we introduce some double sequence spaces using ideal convergence and a sequence of Orlicz functions $\mathcal{M} = (M_{k,l})$ in n -normed spaces and examine some topological properties of the resulting sequence spaces.

KEYWORDS: Paranorm space; I-convergence; Difference sequence spaces; Orlicz function; Musielak-Orlicz function; n -normed spaces.

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1. INTRODUCTION AND PRELIMINARIES

The initial works on double sequences is found in Bromwich [4]. Later on, it was studied by Hardy [13], Moricz [20], Moricz and Rhoades [21], Tripathy ([38, 39]), Başarır and Sonalcan [2] and many others. Hardy [13] introduced the notion of regular convergence for double sequences. Quite recently, Zeltser [41] in her Ph.D thesis has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [25] have recently introduced the statistical convergence and Cauchy convergence for double sequences and given the relation between statistical convergent and strongly Cesaro summable double sequences. Nextly, Mursaleen [23] and Mursaleen and Edely [26] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the M -core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{k,l})$ into one whose core is a subset of the M -core of x . More recently, Altay and Başar [1] have defined the spaces \mathcal{BS} , $\mathcal{BS}(t)$, \mathcal{CS}_p , \mathcal{CS}_{bp} , \mathcal{CS}_r and \mathcal{BV} of double sequences consisting of all double series whose sequence of partial sums are in the spaces \mathcal{M}_u , $\mathcal{M}_u(t)$, \mathcal{C}_p , \mathcal{C}_{bp} , \mathcal{C}_r and \mathcal{L}_u , respectively and also examined some properties of these sequence spaces and determined the α -duals of the spaces \mathcal{BS} , \mathcal{BV} , \mathcal{CS}_{bp} and the $\beta(v)$ -duals

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of the spaces \mathcal{CS}_{bp} and \mathcal{CS}_r of double series. Recently Başar and Sever [3] have introduced the Banach space \mathcal{L}_q of double sequences corresponding to the well known space ℓ_q of single sequences and examined some properties of the space \mathcal{L}_q . Now, recently Raj and Sharma [33] have introduced double sequence spaces of entire functions. By the convergence of a double sequence we mean the convergence in the Pringsheim sense i.e. a double sequence $x = (x_{k,l})$ has Pringsheim limit L (denoted by $P - \lim x = L$) provided that given $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $|x_{k,l} - L| < \epsilon$ whenever $k, l > n$ see [28]. We shall write more briefly as P -convergent. The double sequence $x = (x_{k,l})$ is bounded if there exists a positive number M such that $|x_{k,l}| < M$ for all k and l .

The concept of 2-normed spaces was initially developed by Gähler [8] in the mid of 1960's, while that of n -normed spaces one can see in Misiak [22]. Since then, many others have studied this concept and obtained various results, see Gunawan ([10, 11]) and Gunawan and Mashadi [12] and references therein. Let $n \in \mathbb{N}$ and X be a linear space over the field \mathbb{K} , where \mathbb{K} is field of real or complex numbers of dimension d , where $d \geq n \geq 2$. A real valued function $\|\cdot, \dots, \cdot\|$ on X^n satisfying the following four conditions:

- (i) $\|x_1, x_2, \dots, x_n\| = 0$ if and only if x_1, x_2, \dots, x_n are linearly dependent in X ;
- (ii) $\|x_1, x_2, \dots, x_n\|$ is invariant under permutation;
- (iii) $\|\alpha x_1, x_2, \dots, x_n\| = |\alpha| \|x_1, x_2, \dots, x_n\|$ for any $\alpha \in \mathbb{K}$, and
- (iv) $\|x + x', x_2, \dots, x_n\| \leq \|x, x_2, \dots, x_n\| + \|x', x_2, \dots, x_n\|$

is called a n -norm on X , and the pair $(X, \|\cdot, \dots, \cdot\|)$ is called a n -normed space over the field \mathbb{K} .

For example, we may take $X = \mathbb{R}^n$ being equipped with the Euclidean n -norm $\|x_1, x_2, \dots, x_n\|_E =$ the volume of the n -dimensional parallelepiped spanned by the vectors x_1, x_2, \dots, x_n which may be given explicitly by the formula

$$\|x_1, x_2, \dots, x_n\|_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \dots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \dots, n$. Let $(X, \|\cdot, \dots, \cdot\|)$ be a n -normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \dots, a_n\}$ be linearly independent set in X . Then the following function $\|\cdot, \dots, \cdot\|_\infty$ on X^{n-1} defined by

$$\|x_1, x_2, \dots, x_{n-1}\|_\infty = \max\{\|x_1, x_2, \dots, x_{n-1}, a_i\| : i = 1, 2, \dots, n\}$$

defines an $(n-1)$ -norm on X with respect to $\{a_1, a_2, \dots, a_n\}$.

A sequence (x_k) in a n -normed space $(X, \|\cdot, \dots, \cdot\|)$ is said to converge to some $L \in X$ if

$$\lim_{k \rightarrow \infty} \|x_k - L, z_1, \dots, z_{n-1}\| = 0 \text{ for every } z_1, \dots, z_{n-1} \in X.$$

A sequence (x_k) in a n -normed space $(X, \|\cdot, \dots, \cdot\|)$ is said to be Cauchy if

$$\lim_{\substack{k \rightarrow \infty \\ p \rightarrow \infty}} \|x_k - x_p, z_1, \dots, z_{n-1}\| = 0 \text{ for every } z_1, \dots, z_{n-1} \in X.$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n -norm. Any complete n -normed space is said to be n -Banach space.

The notion of difference sequence spaces was introduced by Kizmaz [14], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [7] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and

$c_0(\Delta^n)$. Let w be the space of all complex or real sequences $x = (x_k)$ and let r be non-negative integers, then for $Z = l_\infty, c, c_0$ we have sequence spaces

$$Z(\Delta^r) = \{x = (x_k) \in w : (\Delta^r x_k) \in Z\},$$

where $\Delta^r x = (\Delta^r x_k) = (\Delta^{r-1} x_k - \Delta^{r-1} x_{k+1})$ and $\Delta^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta^r x_k = \sum_{v=0}^r (-1)^v \binom{r}{v} x_{k+v}.$$

Taking $r = 1$, we get the spaces which were introduced and studied by Kızmaz [14]. An Orlicz function $M : [0, \infty) \rightarrow [0, \infty)$ is a continuous, non-decreasing and convex function such that $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \rightarrow \infty$ as $x \rightarrow \infty$. Lindenstrauss and Tzafriri [17] used the idea of Orlicz function to define the following sequence space,

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \right\}$$

which is called as an Orlicz sequence space. Also ℓ_M is a Banach space with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$

Also, it was shown in [17] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to ℓ_p ($p \geq 1$). The Δ_2 -condition is equivalent to $M(Lx) \leq LM(x)$, for all L with $0 < L < 1$. An Orlicz function M can always be represented in the following integral form

$$M(x) = \int_0^x \eta(t) dt$$

where η is known as the kernel of M , is right differentiable for $t \geq 0$, $\eta(0) = 0$, $\eta(t) > 0$, η is non-decreasing and $\eta(t) \rightarrow \infty$ as $t \rightarrow \infty$.

Let $\lambda = (\lambda_r)$ be a non-decreasing sequence of positive numbers tending to infinity and $\lambda_{r+1} \leq \lambda_r + 1$, $\lambda_1 = 1$. The generalized de la Vallée-Poussin mean is defined by

$$t_r(x) = \frac{1}{\lambda_r} \sum_{k \in I_r} x_k, \quad I_r = [r - \lambda_r + 1, r].$$

A single sequence $x = (x_k)$ is said to be (V, λ) -summable to a number L if $t_r(x) \rightarrow L$ as $r \rightarrow \infty$ see [16]. If $\lambda_r = r$, then the (V, λ) -summability is reduced to $(C, 1)$ -summability see ([36, 37]).

The double sequence $\lambda_2 = (\lambda_{m,n})$ of positive real numbers tending to infinity such that

$$\lambda_{m+1,n} \leq \lambda_{m,n} + 1, \quad \lambda_{m,n+1} \leq \lambda_{m,n} + 1,$$

$$\lambda_{m,n} - \lambda_{m+1,n} \leq \lambda_{m,n+1} - \lambda_{m+1,n+1}, \quad \lambda_{1,1} = 1,$$

and

$$I_{m,n} = \left\{ (k, l) : m - \lambda_{m,n} + 1 \leq k \leq m, \quad n - \lambda_{m,n} + 1 \leq l \leq n \right\}.$$

Let X be a linear metric space. A function $p : X \rightarrow \mathbb{R}$ is called paranorm, if

- (i) $p(x) \geq 0$ for all $x \in X$,
- (ii) $p(-x) = p(x)$ for all $x \in X$,
- (iii) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$,
- (iv) if (λ_n) is a sequence of scalars with $\lambda_n \rightarrow \lambda$ as $n \rightarrow \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \rightarrow 0$ as $n \rightarrow \infty$, then $p(\lambda_n x_n - \lambda x) \rightarrow 0$ as $n \rightarrow \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [40], Theorem 10.4.2, pp. 183). For more details about sequence spaces (see [18, 19, 24, 27, 29-31, 34]) and reference therein.

A sequence space E is said to be solid(or normal) if $(x_k) \in E$ implies $(\alpha_k x_k) \in E$ for all sequences of scalars (α_k) with $|\alpha_k| \leq 1$ and for all $k \in \mathbb{N}$.

The notion of ideal convergence was introduced first by P. Kostyrko [15] as a generalization of statistical convergence which was further studied in topological spaces (see [5]). More applications of ideals can be seen in ([5, 6]).

Recently a lot of activities have started to study sumability, sequence spaces and related topics in these non linear spaces (see [9, 35]). In particular Sahiner [35] combined these two concepts and investigated ideal sumability in these spaces and introduced certain sequence spaces using 2-norm. Raj and Sharma [32] have introduced some sequence spaces of ideal convergence in 2-normed spaces.

We continue in this direction and by using a sequence of Orlicz functions, generalized sequences and also ideals we introduce I-convergence of generalized sequences with respect to a sequence of Orlicz functions in n -normed spaces.

Let $(X, \|\cdot, \dots, \cdot\|)$ be a normed space. Recall that a sequence $(x_n)_{n \in \mathbb{N}}$ of elements of X is called statistically convergent to $x \in X$ if the set $A(\epsilon) = \{n \in \mathbb{N} : \|x_n - x\| \geq \epsilon\}$ has natural density zero for each $\epsilon > 0$.

A family $\mathcal{I} \subset 2^Y$ of subsets of a non empty set Y is said to be an ideal in Y if

- (i) $\phi \in \mathcal{I}$;
- (ii) $A, B \in \mathcal{I}$ imply $A \cup B \in \mathcal{I}$;
- (iii) $A \in \mathcal{I}, B \subset A$ imply $B \in \mathcal{I}$, while an admissible ideal \mathcal{I} of Y further satisfies $\{x\} \in \mathcal{I}$ for each $x \in Y$ (see [9]).

Given $\mathcal{I} \subset 2^{\mathbb{N}}$ be a non trivial ideal in \mathbb{N} . A sequence $(x_n)_{n \in \mathbb{N}}$ in X is said to be I-convergent to $x \in X$, if for each $\epsilon > 0$ the set $A(\epsilon) = \{n \in \mathbb{N} : \|x_n - x\| \geq \epsilon\}$ belongs to \mathcal{I} (see [15]).

Let $\Lambda = (\lambda_{m,n})$ be non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1} \geq \lambda_n + 1$, $\lambda_1 = 0$ and let I be an admissible ideal of \mathbb{N} , $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions, $(X, \|\cdot, \dots, \cdot\|)$ is a n -normed space. Let $p = (p_{k,l})$ be a bounded sequence of positive real numbers. By $S''(n - X)$ we denote the space of all sequences defined over $(X, \|\cdot, \dots, \cdot\|)$. Now we define the following sequence spaces in this paper :

$$(W^I)_2(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \dots, \cdot\|) =$$

$$\left\{ x \in S''(n - X) : \forall \epsilon > 0 \left\{ m, n \in \mathbb{N} : \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \left[M_{k,l} \left(\left\| \frac{\Delta^r x_{k,l} - L}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_{k,l}} \geq \epsilon \right\} \in I \right.$$

$$\left. \text{for some } L, \rho > 0 \text{ and } z_1, \dots, z_{n-1} \in X \right\},$$

$$\begin{aligned}
& (W_0^I)_2(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \dots, \cdot\|) = \\
& \left\{ x \in S''(n-X) : \forall \epsilon > 0 \left\{ m, n \in \mathbb{N} : \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \left[M_{k,l} \left(\left\| \frac{\Delta^r x_{k,l}}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_{k,l}} \geq \epsilon \right\} \in I \right. \\
& \quad \left. \text{for some } \rho > 0 \text{ and } z_1, \dots, z_{n-1} \in X \right\}, \\
& (W_\infty)_2(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \dots, \cdot\|) = \\
& \left\{ x \in S''(n-X) : \exists K > 0 \text{ such that } \sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \left[M_{k,l} \left(\left\| \frac{\Delta^r x_{k,l}}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_{k,l}} \right. \\
& \quad \left. \leq K \text{ for some } \rho > 0 \text{ and } z_1, \dots, z_{n-1} \in X \right\}, \\
& (W_\infty^I)_2(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \dots, \cdot\|) = \\
& \left\{ x \in S''(n-X) : \exists K > 0 \text{ such that } \left\{ m, n \in \mathbb{N} : \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \left[M_{k,l} \left(\left\| \frac{\Delta^r x_{k,l}}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_{k,l}} \right. \right. \\
& \quad \left. \left. \geq K \right\} \in I \text{ for some } \rho > 0 \text{ and } z_1, \dots, z_{n-1} \in X \right\}.
\end{aligned}$$

The following inequality will be used throughout the paper. If $0 \leq p_{k,l} \leq \sup p_{k,l} = H$, $D = \max(1, 2^{H-1})$ then

$$|a_{k,l} + b_{k,l}|^{p_{k,l}} \leq D\{|a_{k,l}|^{p_{k,l}} + |b_{k,l}|^{p_{k,l}}\} \quad (1.1)$$

for all k, l and $a_{k,l}, b_{k,l} \in \mathbb{C}$. Also $|a|^{p_{k,l}} \leq \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

The main aim of this paper is to study some topological properties and some inclusion relation between above defined sequence spaces.

2. MAIN RESULTS

Theorem 2.1. *Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions, $p = (p_{k,l})$ be a bounded sequence of positive real numbers and I be an admissible ideal of \mathbb{N} . Then $(W^I)_2(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \dots, \cdot\|)$, $(W_0^I)_2(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \dots, \cdot\|)$, $(W_\infty)_2(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \dots, \cdot\|)$ and $(W_\infty^I)_2(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \dots, \cdot\|)$ are linear spaces.*

Proof. Let $x = (x_{k,l}), y = (y_{k,l}) \in (W^I)_2(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \dots, \cdot\|)$ and $\alpha, \beta \in \mathbb{R}$. So

$$\begin{aligned}
& \left\{ m, n \in \mathbb{N} : \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \left[M_{k,l} \left(\left\| \frac{\Delta^r x_{k,l} - L}{\rho_1}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_{k,l}} \geq \epsilon \right\} \in I \\
& \quad \text{for some } L, \rho_1 > 0, \text{ and } z_1, \dots, z_{n-1} \in X \}
\end{aligned}$$

and

$$\begin{aligned}
& \left\{ m, n \in \mathbb{N} : \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \left[M_{k,l} \left(\left\| \frac{\Delta^r y_{k,l} - L}{\rho_2}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_{k,l}} \geq \epsilon \right\} \in I \\
& \quad \text{for some } L, \rho_2 > 0 \text{ and } z_1, \dots, z_{n-1} \in X \}.
\end{aligned}$$

Since $\|\cdot, \dots, \cdot\|$ is a n -norm and $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions the following inequality holds:

$$\begin{aligned}
& \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \left[M_{k,l} \left(\left\| \frac{\Delta^r (\alpha x_{k,l} + \beta y_{k,l}) - L}{|\alpha|\rho_1 + |\beta|\rho_2}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_{k,l}} \\
& \leq D \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \left[\frac{|\alpha|}{(|\alpha|\rho_1 + |\beta|\rho_2)} M_{k,l} \left(\left\| \frac{\Delta^r x_{k,l} - L}{\rho_1}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_{k,l}}
\end{aligned}$$

$$\begin{aligned}
& + D \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \left[\frac{|\beta|}{(|\alpha|\rho_1 + |\beta|\rho_2)} M_{k,l} \left(\left\| \frac{\Delta^r y_{k,l} - L}{\rho_2}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_{k,l}} \\
& \leq DF \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \left[M_{k,l} \left(\left\| \frac{\Delta^r x_{k,l} - L}{\rho_1}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_{k,l}} \\
& + DF \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \left[M_{k,l} \left(\left\| \frac{\Delta^r y_{k,l} - L}{\rho_2}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_{k,l}},
\end{aligned}$$

where $F = \max \left[1, \left(\frac{|\alpha|}{(|\alpha|\rho_1 + |\beta|\rho_2)} \right)^H, \left(\frac{|\beta|}{(|\alpha|\rho_1 + |\beta|\rho_2)} \right)^H \right]$. From the above inequality, we get $\left\{ m, n \in \mathbb{N} : \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \left[M_{k,l} \left(\left\| \frac{\Delta^r (\alpha x_{k,l} + \beta y_{k,l}) - L}{|\alpha|\rho_1 + |\beta|\rho_2}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_{k,l}} \geq \epsilon \right\}$

$$\begin{aligned}
& \subseteq \left\{ m, n \in \mathbb{N} : DF \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \left[M_{k,l} \left(\left\| \frac{\Delta^r x_{k,l} - L}{\rho_1}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_{k,l}} \geq \frac{\epsilon}{2} \right\} \\
& \cup \left\{ m, n \in \mathbb{N} : DF \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \left[M_{k,l} \left(\left\| \frac{\Delta^r y_{k,l} - L}{\rho_2}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_{k,l}} \geq \frac{\epsilon}{2} \right\}.
\end{aligned}$$

Two sets on the right hand side belong to I and this completes the proof. Similarly, we can prove that $(W_0^I)_2(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \dots, \cdot\|)$, $(W_\infty)_2(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \dots, \cdot\|)$ and $(W_\infty^I)_2(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \dots, \cdot\|)$ are linear spaces. \square

Theorem 2.2. Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions and $p = (p_{k,l})$ be a bounded sequence of positive real numbers. For any fixed $m, n \in \mathbb{N}$, $(W_\infty)_2(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \dots, \cdot\|)$ is a paranormed space with the paranorm defined by

$$\begin{aligned}
g_{m,n}(x) &= \inf \left\{ \rho^{\frac{p_{m,n}}{H}} : \rho > 0 \text{ is such that } \sup_{k,l} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \left[M_{k,l} \left(\left\| \frac{\Delta^r x_{k,l}}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_{k,l}} \right. \\
& \quad \left. \leq 1, \forall z_1, \dots, z_{n-1} \in X \right\}.
\end{aligned}$$

Proof. It is clear that $g_{m,n}(x) = g_{m,n}(-x)$. Since $M_{k,l}(0) = 0$, we get $\inf \left\{ \rho^{\frac{p_{m,n}}{H}} \right\} = 0$ for $x = 0$ therefore, $g_{m,n}(0) = 0$. Let us take $x = (x_{k,l})$ and $y = (y_{k,l})$ in $(W_\infty)_2(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \dots, \cdot\|)$. Let

$$B(x) = \left\{ \rho > 0 : \sup_{k,l} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \left[M_{k,l} \left(\left\| \frac{\Delta^r x_{k,l}}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_{k,l}} \leq 1, \forall z_1, \dots, z_{n-1} \in X \right\},$$

$$B(y) = \left\{ \rho > 0 : \sup_{k,l} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \left[M_{k,l} \left(\left\| \frac{\Delta^r y_{k,l}}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_{k,l}} \leq 1, \forall z_1, \dots, z_{n-1} \in X \right\}.$$

Let $\rho_1 \in B(x)$ and $\rho_2 \in B(y)$. Then if $\rho = \rho_1 + \rho_2$, we have

$$\begin{aligned}
& \sup_{k,l} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} M_{k,l} \left(\left\| \frac{\Delta^r (x_{k,l} + y_{k,l})}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \\
& \leq \frac{\rho_1}{\rho_1 + \rho_2} \sup_{k,l} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} M_{k,l} \left(\left\| \frac{\Delta^r x_{k,l}}{\rho_1}, z_1, \dots, z_{n-1} \right\| \right)
\end{aligned}$$

$$+ \frac{\rho_2}{\rho_1 + \rho_2} \sup_{k,l} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} M_{k,l} \left(\left\| \frac{\Delta^r y_{k,l}}{\rho_2}, z_1, \dots, z_{n-1} \right\| \right).$$

Thus $\sup_{k,l} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} M_{k,l} \left(\left\| \frac{\Delta^r (x_{k,l} + y_{k,l})}{\rho_1 + \rho_2}, z_1, \dots, z_{n-1} \right\| \right)^{p_{k,l}} \leq 1$ and

$$\begin{aligned} g_{m,n}(x+y) &\leq \inf \left\{ (\rho_1 + \rho_2)^{\frac{p_{m,n}}{H}} : \rho_1 \in B(x), \rho_2 \in B(y) \right\} \\ &\leq \inf \left\{ \rho_1^{\frac{p_{m,n}}{H}} : \rho_1 \in B(x) \right\} + \inf \left\{ \rho_2^{\frac{p_{m,n}}{H}} : \rho_2 \in B(y) \right\} \\ &= g_{m,n}(x) + g_{m,n}(y). \end{aligned}$$

Let $\sigma^s \rightarrow \sigma$ where $\sigma, \sigma^s \in \mathbb{C}$ and let $g_{m,n}(x_{k,l}^s - x) \rightarrow 0$ as $s \rightarrow \infty$.

We have to show that $g_{m,n}(\sigma^s x_{k,l}^s - \sigma x) \rightarrow 0$ as $s \rightarrow \infty$. Let

$$B(x^s) = \left\{ \rho_s > 0 : \sup_{k,l} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \left[M_{k,l} \left(\left\| \frac{\Delta^r x_{k,l}^s}{\rho_s}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_{k,l}} \leq 1, \forall z_1, \dots, z_{n-1} \in X \right\},$$

$$\begin{aligned} B(x^s - x) &= \left\{ \rho'_s > 0 : \sup_{k,l} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \left[M_{k,l} \left(\left\| \frac{\Delta^r x_{k,l}^s - x_{k,l}}{\rho'_s}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_{k,l}} \leq 1, \right. \\ &\quad \left. \forall z_1, \dots, z_{n-1} \in X \right\}. \end{aligned}$$

If $\rho_s \in B(x^s)$ and $\rho'_s \in B(x^s - x)$ then we observe that

$$\begin{aligned} &\frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} M_{k,l} \left\| \frac{\Delta^r (\sigma^s x_{k,l}^s - \sigma x_{k,l})}{\rho_s |\sigma^s - \sigma| + \rho'_s |\sigma|}, z_1, \dots, z_{n-1} \right\| \\ &\leq \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} M_{k,l} \left(\left\| \frac{\Delta^r (\sigma^s x_{k,l}^s - \sigma x_{k,l}^s)}{\rho_s |\sigma^s - \sigma| + \rho'_s |\sigma|}, z_1, \dots, z_{n-1} \right\| \right) \\ &\quad + \left\| \frac{\Delta^r (\sigma x_{k,l}^s - \sigma x_{k,l})}{\rho_s |\sigma^s - \sigma| + \rho'_s |\sigma|}, z_1, \dots, z_{n-1} \right\| \\ &\leq \frac{|\sigma^s - \sigma| \rho_s}{\rho_s |\sigma^s - \sigma| + \rho'_s |\sigma|} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} M_{k,l} \left(\left\| \frac{\Delta^r (x_{k,l}^s)}{\rho_s}, z_1, \dots, z_{n-1} \right\| \right) \\ &\quad + \frac{|\sigma| \rho'_s}{\rho_s |\sigma^s - \sigma| + \rho'_s |\sigma|} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} M_{k,l} \left(\left\| \frac{\Delta^r (x_{k,l}^s - x_{k,l})}{\rho'_s}, z_1, \dots, z_{n-1} \right\| \right). \end{aligned}$$

From the above inequality, it follows that

$$\frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \left(M_{k,l} \left(\left\| \frac{\Delta^r (\sigma^s x_{k,l}^s - \sigma x_{k,l})}{\rho_s |\sigma^s - \sigma| + \rho'_s |\sigma|}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_{k,l}} \leq 1$$

and consequently,

$$\begin{aligned} g_{m,n}(\sigma^s x^s - \sigma x) &\leq \inf \left\{ \left(\rho_s |\sigma^s - \sigma| + \rho'_s |\sigma| \right)^{\frac{p_{m,n}}{H}} : \rho_s \in B(x^s), \rho'_s \in B(x^s - x) \right\} \\ &\leq (|\sigma^s - \sigma|)^{\frac{p_{m,n}}{H}} \inf \left\{ \rho^{\frac{p_{m,n}}{H}} : \rho \in B(x^s) \right\} \\ &\quad + (|\sigma|)^{\frac{p_{m,n}}{H}} \inf \left\{ (\rho'_s)^{\frac{p_{m,n}}{H}} : \rho'_s \in B(x^s - x) \right\} \\ &\longrightarrow 0 \quad \text{as } m \longrightarrow \infty. \end{aligned}$$

This completes the proof. \square

Theorem 2.3. Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions which satisfies Δ_2 -condition. Then $(W_0^I)_2(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \dots, \cdot\|) \subset (W^I)_2(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \dots, \cdot\|) \subset (W_\infty^I)_2(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \dots, \cdot\|)$ and the inclusions are strict.

Proof. The inclusion $(W_0^I)_2(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \dots, \cdot\|) \subset (W^I)_2(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \dots, \cdot\|)$ is obvious. We have only show that $(W^I)_2(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \dots, \cdot\|) \subset (W_\infty^I)_2(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \dots, \cdot\|)$.

Let $(x_{k,l}) \in (W^I)_2(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \dots, \cdot\|)$. Then

$$\begin{aligned} & \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \left[M_{k,l} \left(\left\| \frac{\Delta^r x_{k,l}}{2\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_{k,l}} \\ &= \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \left[M_{k,l} \left(\left\| \frac{\Delta^r x_{k,l} + L - L}{2\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_{k,l}} \\ &\leq \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \left[M_{k,l} \left(\left\| \frac{\Delta^r x_{k,l} - L}{2\rho}, z_1, \dots, z_{n-1} \right\| + \left\| \frac{L}{2\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_{k,l}} \\ &\leq DG \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \left[M_{k,l} \left(\left\| \frac{\Delta^r x_{k,l} - L}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_{k,l}} \\ &+ DG \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \left[M_{k,l} \left(\left\| \frac{L}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_{k,l}}, \end{aligned}$$

where $G = \max \left\{ 1, \left(\frac{1}{2} \right)^H \right\}$. Thus from Δ_2 -condition, we have $x \in (W_\infty^I)_2(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \dots, \cdot\|)$ and this completes the proof of the theorem. \square

Theorem 2.4. Let $\mathcal{M}, \mathcal{M}', \mathcal{M}''$ are sequences of Orlicz functions. Then we have

(i) $(W_0^I)_2(\lambda, \mathcal{M}', \Delta^r, p, \|\cdot, \dots, \cdot\|) \subseteq (W_0^I)_2(\lambda, \mathcal{M} \circ \mathcal{M}', \Delta^r, p, \|\cdot, \dots, \cdot\|)$ provided $(p_{k,l})$ is such that $H_0 = \inf p_{k,l} > 0$.

(ii) $(W_0^I)_2(\lambda, \mathcal{M}', \Delta^r, p, \|\cdot, \dots, \cdot\|) \cap (W_0^I)_2(\lambda, \mathcal{M}'', \Delta^r, p, \|\cdot, \dots, \cdot\|) \subseteq (W_0^I)_2(\lambda, \mathcal{M}' + \mathcal{M}'', \Delta^r, p, \|\cdot, \dots, \cdot\|)$.

Proof. (i) For given $\epsilon > 0$, first choose $\epsilon_0 > 0$ such that $\max\{\epsilon_0^H, \epsilon_0^{H_0}\} < \epsilon$. Now using the continuity of $M_{k,l}$. Choose $0 < \delta < 1$ such that $0 < t < \delta$, this implies that $M_{k,l}(t) < \epsilon_0$. Let $(x_{k,l}) \in (W_0^I)_2(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \dots, \cdot\|)$. Now from the definition

$$B(\delta) = \left\{ m, n \in \mathbb{N} : \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \left[M'_{k,l} \left(\left\| \frac{\Delta^r x_{k,l}}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_{k,l}} \geq \delta^H \right\} \in I.$$

Thus if $m, n \notin B(\delta)$ then

$$\begin{aligned} & \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \left[M'_{k,l} \left(\left\| \frac{\Delta^r x_{k,l}}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_{k,l}} < \delta^H \\ & \implies \sum_{k,l \in I_{m,n}} \left[M'_{k,l} \left(\left\| \frac{\Delta^r x_{k,l}}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_{k,l}} < \lambda_{m,n} \delta^H \\ & \implies \left[M'_{k,l} \left(\left\| \frac{\Delta^r x_{k,l}}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_{k,l}} < \delta^H \text{ for all } k, l \in I_{m,n} \\ & \implies \left[M'_{k,l} \left(\left\| \frac{\Delta^r x_{k,l}}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right] < \delta \text{ for all } k, l \in I_{m,n}. \end{aligned}$$

Hence from above using the continuity of $\mathcal{M} = (M_{k,l})$ we must have

$$M_{k,l} \left(M'_{k,l} \left(\left\| \frac{\Delta^r x_{k,l}}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right) < \epsilon_0 \quad \forall k, l \in I_{m,n}$$

which consequently implies that

$$\begin{aligned} \sum_{k,l \in I_{m,n}} \left[M_{k,l} \left(M'_{k,l} \left(\left\| \frac{\Delta^r x_{k,l}}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right) \right]^{p_{k,l}} &< \lambda_{m,n} \max\{\epsilon_0^H, \epsilon_0^{H_0}\} \\ &< \lambda_{m,n} \epsilon \\ \implies \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \left[M_{k,l} \left(M'_{k,l} \left(\left\| \frac{\Delta^r x_{k,l}}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right) \right]^{p_{k,l}} &< \epsilon. \end{aligned}$$

This shows that

$$\left\{ m, n \in \mathbb{N} : \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \left[M_{k,l} \left(M'_{k,l} \left(\left\| \frac{\Delta^r x_{k,l}}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right) \right]^{p_{k,l}} \geq \epsilon \right\} \subset B(\delta)$$

and so belongs to I . This proves the result.

(ii) Let $(x_{k,l}) \in (W_0^I)_2(\lambda, \mathcal{M}', \Delta^r, p, \|\cdot, \dots, \cdot\|) \cap (W_0^I)_2(\lambda, \mathcal{M}''_{k,l}, \Delta^r, p, \|\cdot, \dots, \cdot\|)$. Then the fact

$$\begin{aligned} \frac{1}{\lambda_{m,n}} \left[(M'_{k,l} + M''_{k,l}) \left(\left\| \frac{\Delta^r x_{k,l}}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_{k,l}} \\ \leq D \frac{1}{\lambda_{m,n}} \left[M'_{k,l} \left(\left\| \frac{\Delta^r x_{k,l}}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_{k,l}} \\ + D \frac{1}{\lambda_{m,n}} \left[M''_{k,l} \left(\left\| \frac{\Delta^r x_{k,l}}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_{k,l}} \end{aligned}$$

gives the result. \square

Theorem 2.5. *The sequence spaces $(W_0^I)_2(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \dots, \cdot\|)$ and $(W_\infty^I)_2(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \dots, \cdot\|)$ are solid.*

Proof. Let $(x_k) \in (W_0^I)_2(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \dots, \cdot\|)$, let $(\alpha_{k,l})$ be a sequence of scalars such that $|\alpha_{k,l}| \leq 1$ for all $k, l \in \mathbb{N}$. Then we have

$$\begin{aligned} \left\{ m, n \in \mathbb{N} : \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \left[M_{k,l} \left(\left\| \frac{\Delta^r \alpha_{k,l} x_{k,l}}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_{k,l}} \right\} \subset \\ \left\{ m, n \in \mathbb{N} : \frac{C}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \left[M_{k,l} \left(\left\| \frac{\Delta^r x_{k,l}}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_{k,l}} \geq \epsilon \right\} \in I, \end{aligned}$$

where $C = \max\{1, |\alpha_{k,l}|^H\}$. Hence $(\alpha_{k,l} x_{k,l}) \in (W_0^I)_2(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \dots, \cdot\|)$ for all sequences of scalars $\alpha_{k,l}$ with $|\alpha_{k,l}| \leq 1$ for all $k, l \in \mathbb{N}$ whenever $(x_{k,l}) \in (W_0^I)_2(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \dots, \cdot\|)$.

Similarly, we can prove that $(W_\infty^I)_2(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \dots, \cdot\|)$ is also solid. \square

Theorem 2.6. *The sequence spaces $(W_0^I)_2(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \dots, \cdot\|)$ and $(W_\infty^I)_2(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \dots, \cdot\|)$ are monotone.*

Proof. It is easy to prove so we omit the details. \square

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