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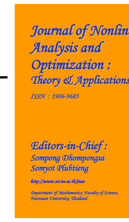

BEST APPROXIMATION FOR CONVEX SUBSETS OF 2-INNER PRODUCT SPACES

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ABSTRACT. In this paper, we study the concept of best approximation in 2-inner product spaces. We get some characteristic theorems for the elements of best approximation for convex subsets of 2-inner product spaces. Finally we get some properties of the metric projection map in this spaces.

KEYWORDS : 2-Inner product space; 2-Normed space; b-Best approximation; b-Proximinal; b-Chebyshev; b-Metric projection; b-Dual cone
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1. INTRODUCTION

Recently, some results on best approximation theory in linear 2-normed spaces have been obtained by Y. J. Cho, S. Elumalai, S. S. Kim, R. Ravi, Sh. Rezapour and others (see [1], [4], [6], [12], [13], [15]). These papers are based on the research works in normed linear spaces made by I. Singer ([14]), T. D. Narang ([11]), S. S. Dragomir ([2]) and others. In this paper we want to investigate the concept of best approximation in 2-inner product spaces. The concept of 2-inner product spaces has been investigated by R. Ehret in 1969([3]), and has been developed extensively in different subjects by others.([10], [8])

Definition 1.1. Let X be a linear space of dimension greater than 1 over filed $\mathbb{R}$ of real numbers.
Suppose that $\langle .,.|.\rangle$ is a $\mathbb{R}$-valued function defined on $X \times X \times X$ satisfying the following conditions:
a) $\langle x,x|z\rangle \geq 0$ and $\langle x,x|z\rangle = 0$ if and only if x and z are linearly dependent.
b) $\langle x,x|z\rangle = \langle z,z|x\rangle$
c) $\langle y,x|z\rangle = \langle x,y|z\rangle$
d) $\langle \alpha x,y|z\rangle = \alpha\langle x,y|z\rangle$ for any scalar $\alpha \in \mathbb{R}$
e) $\langle x+x',y|z\rangle = \langle x,y|z\rangle + \langle x',y|z\rangle$

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$\langle .,.|. \rangle$ is called a *2-inner product* and $(X, \langle .,.|. \rangle)$ is called a *2-inner product space*(or a *2-per-Hilbert space*). A concept which is closely related to 2-inner product space and introduced by Gähler in 1965, is 2-normed space [5].

Definition 1.2. Let X be a linear space of dimension greater than 1 over filed $\mathbb{R}$ of real numbers. Suppose $\|.,.\|$is a real-valued function on $X \times X$ satisfying the following conditions:
a) $\|x, y\| = 0$ if and only if x and y are linearly dependent vectors.
b) $\|x, y\| = \|y, x\|$ for all $x, y \in X$.
c) $\|\lambda x, y\| = |\lambda|\|x, y\|$ for all $\lambda \in \mathbb{R}$ and $x, y \in X$.
d) $\|x + y, z\| \leq \|x, z\| + \|y, z\|$ for all $x, y, z \in X$.

Then $\|.,.\|$ is called a *2-norm* on X and $(X, \|.,.\|)$ is called a *linear 2-normed space*. It is easy to show that the 2-norm $\|.,.\|$ is non-negative and $\|x, y + \alpha x\| = \|x, y\|$ for all $x, y \in X$ and $\alpha \in \mathbb{R}$. Every 2-normed space is a locally convex topological vector space. In fact for a fixed $b \in X, p_b(x) = \|x, b\|; x \in X$ is a semi-norm on X and the family $P = \{p_b : b \in X\}$ of seminorms generates a locally convex topology on X.
Let $(X, \langle .,.|. \rangle)$ be a 2-inner product space, then
(i) We can define a 2-norm on $X \times X$ by $\|x, y\| = \sqrt{\langle x, x|y \rangle}$.
(ii) Let $0 \neq b \in X$ and $x, y \in X \setminus < b >$. An element $x \in X$ is said to be *b-orthogonal* to an element $y \in X$, and we write $x \perp_b y$, if $\langle x, y|b \rangle = 0$.
(iii) For all $x, y, b \in X$, we have the Cauchy-Schwartz inequality

$$\langle x, y|b \rangle^2 \leq \|x, b\|^2 \|y, b\|^2.$$

Let $(X, \|.,.\|)$ be a 2-normed space and V_1 and V_2 be two linear subspaces of X. A 2-functional $f : V_1 \times V_2 \to \mathbb{R}$ is called a bilinear 2-functional on $V_1 \times V_2$, whenever for all $x_1, x_2 \in V_1, y_1, y_2 \in V_2$ and $\lambda_1, \lambda_2 \in \mathbb{R}$;
i) $f(x_1 + x_2, y_1 + y_2) = f(x_1, y_1) + f(x_1, y_2) + f(x_2, y_1) + f(y_1, y_2)$,
ii) $f(\lambda_1 x_1, \lambda_2 y_1) = \lambda_1 \lambda_2 f(x_1, y_1)$.
A bilinear 2-functional $f : V_1 \times V_2 \to \mathbb{R}$ is said to be bounded if there exists a non-negative real number M (called a Lipschitz constant for f) such that $|f(x, y)| \leq M\|x, y\|$ for all $x \in V_1$ and $y \in V_2$. Also, the norm of a bilinear 2-functional f is defined by

$$\|f\| = \inf\{M \geq 0 : M \text{ is a Lipschitz constant for } f\}$$

It is known that

$$\begin{aligned}\|f\| &= \sup\{|f(x, y)| : (x, y) \in V_1 \times V_2, \|x, y\| \leq 1\} \\ &= \sup\{|f(x, y)| : (x, y) \in V_1 \times V_2, \|x, y\| = 1\} \\ &= \sup\{|f(x, y)|/\|x, y\| : (x, y) \in V_1 \times V_2, \|x, y\| > 0\}.\end{aligned}$$

Definition 1.3. [7] A 2-functional $F : V_1 \times V_2 \to \mathbb{R}$ is said to be a convex 2-functional if

$$\begin{aligned}F(a\lambda x + (a - a\lambda)x', b\mu y + (b - b\mu)y') \leq ab|\lambda\mu|F(x, y) + a|\lambda|(b - b\mu)F(x, y') \\ + (a - a\lambda)b|\mu|F(x', y) + (a - a\lambda)(b - b\mu)F(x', y')\end{aligned}$$

for all $|\lambda| \leq 1, |\mu| \leq 1$ and $a, b \geq 0$.

Definition 1.4. Let $(X, \langle .,.|. \rangle)$ be a 2-inner product space, and $b \in X$.

1) A sequence $\{x_n\}$ of X is said to be *b-convergent* and denote by $x_n \xrightarrow{b} x$, if there exists an element $x \in X$ such that $\lim_{n \to \infty} \|x_n - x, b\| = 0$ for all $x \in X$.

2) A subset E of X is said *b-closed*, if for each sequence $\{x_n\}$ in E such that $x_n \xrightarrow{b} x$, we have that $x \in E$.

Definition 1.5. [13] Let $(X, \|.,.\|)$ be a 2-normed space, G a nonempty subset of X, $0 \neq b \in X$, then $g_0 \in G$ is called a *b-best approximation* to x from G if

$$\|x - g_0, b\| = \inf\{\|x - g, b\| : g \in G\}.$$

The set of all b-best approximations of x in G is denoted by $P_{G,b}(x)$. The mapping $P_{G,b} : X \longrightarrow 2^G$ is called the *b-metric projection* onto G.
If each $x \in X \setminus (G + < b >)$ has at least (resp. exactly) one b-best approximation in G, then G is called a *b-proximinal* (resp. *b-chebyshev*) set.

Definition 1.6. 1) A nonempty subset K of the 2-inner product space X, is called convex if $\lambda x + (1 - \lambda)y \in K$ whenever $x, y \in K$ and $0 \leq \lambda \leq 1$.
2) A nonempty subset C of the 2-inner product space X, is called a convex cone if $\alpha x + \beta y \in C$ whenever $x, y \in C$ and $0 \leq \alpha, \beta \in \mathbb{R}$.
3) A nonempty subset M of the 2-inner product space X, is called a linear subspace if $\alpha x + \beta y \in M$ whenever $x, y \in M$ and $\alpha, \beta \in \mathbb{R}$.
4) A nonempty subset V of the 2-inner product space X, is called affine if $\alpha x + (1 - \alpha)y \in V$ whenever $x, y \in V$, and $\alpha \in \mathbb{R}$.

Definition 1.7. Let $(X, \langle ., .|.\rangle)$ be a 2-inner product space over the real number field $\mathbb{R}$. The *b-dual cone* (or *b-negative polar*) of S is the set

$$S_b^\circ := \{x \in X | \langle x, y|b\rangle \leq 0 \text{ for all } y \in S\}$$

The *b-orthogonal complement* of S is the set

$$S_b^\perp = S_b^\circ \cap (-S_b^\circ) = \{x \in X | \langle x, y|b\rangle = 0 \text{ for all } y \in S\}$$

2. CHARACTERIZATION THEOREMS FOR ELEMENTS OF B-BEST APPROXIMATION FOR CONVEX SUBSETS OF A 2-INNER PRODUCT SPACE

In this section we investigate some characteristic theorems for elements of b-best approximation for convex subsets of a 2-inner product space X. It is known that every nonempety b-closed convex set in a b-Hilbert space is b-chebyshev(see [9], [15]). Now we have the following theorem.

Theorem 2.1. *Let $(X, \langle ., .|.\rangle)$ be a 2-inner product space, and K a b-closed convex subset of X with $X \neq K$. If $x_0 \in X \setminus (K + < b >)$ and $g_0 \in K$, then the following statements are equivalent.*
(1) $g_0 = P_{K,b}(x_0)$
(2) $x_0 - g_0 \in (K - g_0)_b^\circ$
(3) we have that

$$\inf_{g \in K} \langle g - x_0, g_0 - x_0|b\rangle = \|g_0 - x_0, b\|^2$$

Proof. $(1) \Rightarrow (2)$: This implication is proved in ([15]).
$(2) \Rightarrow (3)$: By (2) we have that for all $g \in K$,

$$\begin{aligned} 0 \leq \langle x_0 - g_0, g_0 - g|b\rangle &= \langle x_0 - g_0, g_0 - x_0 + x_0 - g|b\rangle \\ &= \langle x_0 - g_0, x_0 - g|b\rangle - \langle x_0 - g_0, x_0 - g_0|b\rangle \\ &= \langle x_0 - g_0, x_0 - g|b\rangle - \|x_0 - g_0, b\|^2. \end{aligned}$$

Hence

$$\|x_0 - g_0, b\|^2 \leq \langle x_0 - g_0, x_0 - g|b\rangle$$

for all $g \in K$, Thus

$$\begin{aligned}\|x-g_0,b\|^2 &\le \inf_{g\in K}\langle x_0-g_0,x_0-g|b\rangle\\ &\le \langle x_0-g_0,x_0-g_0|b\rangle = \|x-g_0,b\|^2\end{aligned}$$

Therefore

$$\|x_0-g_0,b\|^2 = \inf_{g\in K}\langle x_0-g_0,x_0-g|b\rangle.$$

$(3) \Rightarrow (1)$: If (3) holds, then by Cauchy-Schwartz inequality in 2-inner product spaces, for all $g \in K$ we have

$$\|x_0-g_0,b\|^2 \le \langle x_0-g_0,x_0-g|b\rangle \le \|x_0-g_0,b\|\|x_0-g,b\|.$$

Thus $\|x_0-g_0,b\| \le \|x_0-g,b\|$ for all $g \in K$. That is $g_0 = P_{K,b}(x_0)$.

□

Lemma 2.1. *Let X be a 2-inner product space over the real field $\mathbb{R}$. Then:*
(1) If S is a nonempty subset of X, then S_b° is a b-closed convex cone and $S_b^\perp$ is a b-closed subspace.
(2) If C is a convex cone in X, then $(C-y)_b^\circ = C_b^\circ \cap y_b^\perp$ for each $y \in C$.
(3) If M is a subspace of X, then $M_b^\circ = M_b^\perp$.
(4) If C is a b-chebyshev convex cone in X, then $C_b^{\circ\circ} = C$.
(5) If M is a b-chebyshev subspace in X, then $M_b^{\circ\circ} = M_b^{\perp\perp} = M$.

Proof. (1) Let $x_n \in S_b^\circ$ and $x_n \xrightarrow{b} x$. Then for each $y \in S$,

$$\langle x,y|b\rangle = \lim\langle x_n,y|b\rangle \le 0$$

implies $x \in S_b^\circ$ and S_b° is b-closed. Let $x,z \in S_b^\circ$ and $\alpha,\beta \ge 0$. Then, for each $y \in S$,

$$\langle \alpha x+\beta z,y|b\rangle = \alpha\langle x,y|b\rangle + \beta\langle z,y|b\rangle \le 0$$

so $\alpha x+\beta z \in S_b^\circ$ and S_b° is a convex cone. Similarly we can prove $S_b^\perp$ is a b-closed subspace.
(2) We have $x \in (C-y)_b^\circ$ if and only if $\langle x,c-y|b\rangle \le 0$ for all $c \in C$. Taking $c=0$ and $c=2y$, it follows that the last statement is equivalent to $\langle x,y|b\rangle = 0$ and $\langle x,c|b\rangle \le 0$ for all $c \in C$. That is, $x \in C_b^\circ \cap y_b^\perp$.
(3) If M is a subspace, then $-M = M$ implies

$$M_b^\circ = M_b^\circ \cap (-M)_b^\circ = M_b^\perp.$$

(4) Let C be a b-chebyshev convex cone, and $x \in C\backslash <b>$. Then for any $y \in C_b^\circ, \langle x,y|b\rangle \le 0$. Hence $x \in C_b^{\circ\circ}$. That is $C \subseteq C_b^{\circ\circ}$. Now remains to verify $C_b^{\circ\circ} \subseteq C$. If not, choose $x \in C_b^{\circ\circ}\backslash C$ and let $y_0 \in P_{C,b}(x)$. By (2) and theorem (2.1) we have

$$x-y_0 \in (C-y_0)_b^\circ = C_b^\circ \cap y_{0_b}^\perp.$$

Thus

$$0 < \|x-y_0,b\|^2 = \langle x-y_0,x-y_0|b\rangle = \langle x-y_0,x|b\rangle \le 0$$

which is absurd. Therefore $C_b^{\circ\circ} \subseteq C$.
(5) It is clear by (3),(4). □

Now we investigate theorem (2.1) in the case of b-closed convex cone.

Theorem 2.2. *Let $(X,\langle.,.|.\rangle)$ be a 2-inner product space, C a b-closed convex cone in X with $X \ne C$. If $x_0 \in X\backslash(C+<b>)$ and $g_0 \in C$, then the following statements are equivalent:*
(1) $g_0 = P_{C,b}(x_0)$,

(2) $x_0 - g_0 \in C_b^\circ \cap {g_0}_b^\perp$,
(3) $x_0 - g_0 \in C_b^\circ$ *and* $\langle x_0, g_0|b\rangle = \|g_0, b\|^2$

Proof. The equivalence of (2) and (3) is clear. The equivalence of (1) and (2)is also clear by lemma (2.2). □

Remark 2.2. If M is a linear subspace then by lemma(2.2), $M_b^\circ = M_b^\perp$ and so condition (2) of above theorem reduces to the classical condition that

$$g_0 = P_{M,b}(x_0) \Longleftrightarrow x - g_0 \perp_b M \Longleftrightarrow \langle x - g_0, g|b\rangle = 0 \text{ for all } g \in M \text{ ; (see [13]).}$$

Remark 2.3. Let V be an affine set in the 2-inner product space X, i.e. $V = M + v$, where M is a subspace and v is any element of V. Then we have that

$$g_0 = P_{V,b}(x_0) \Longleftrightarrow x_0 - g_0 \perp_b M \Longleftrightarrow \langle x_0 - g_0, g - v|b\rangle = 0 \text{ for all } g \in V.$$

Moreover

$$P_{V,b}(x + e) = P_{V,b}(x) \text{ for all } x \in X, e \in M_b^\perp.$$

If $(X, \langle ., .\rangle)$ is an inner product space, then the function

$$\langle x, y|z\rangle = \begin{vmatrix} \langle x, y\rangle & \langle x, z\rangle \\ \langle y, z\rangle & \langle z, z\rangle \end{vmatrix} = \|z\|^2\langle x, y\rangle - \langle x, z\rangle\langle y, z\rangle$$

for all $x, y, z \in X$ defined a 2-inner product on $X \times X \times X$.

Example 2.4. Let $X = \mathbb{R}^2$, $C = \{(y_1, y_2) \in \mathbb{R}^2; |y_2| \leq y_1\}$ and $b = (b_1, b_2) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. It is simple to verify that C is a b-closed convex cone. Now if $(x_1, x_2) \in C+ < b >$, then

$$P_{C,b}(x_1, x_2) = \begin{cases} \big(\{\frac{x_2b_1 - x_1b_2}{b_2 - b_1}, \frac{x_2b_1 - x_1b_2}{b_2 - b_1}\} + < b > \big) \bigcap C & \text{if } x_2 > 0, b_1 \neq b_2 \\ \big((x_1, x_2) + < b > \big) \bigcap C & \text{if } b_1 = b_2 \\ \big(\{\frac{x_2b_1 - x_1b_2}{b_2 + b_1}, \frac{x_1b_2 - x_2b_1}{b_2 + b_1}\} + < b > \big) \bigcap C & \text{if } x_2 < 0, b_1 \neq -b_2 \\ \big((x_1, x_2) + < b > \big) \bigcap C & \text{if } b_1 = -b_2 \end{cases}$$

and if $(x_1, x_2) \in \mathbb{R}^2 \setminus (C+ < b >)$, then $P_{C,b}(x_1, x_2) = \{(0, 0)\}$.

Let $(X, \langle ., .|.\rangle)$ be a 2-inner product space over the real field $\mathbb{R}$ and let $F : X\times < b > \longrightarrow \mathbb{R}$ be a continues convex 2-functional on $X\times < b >$. Denote by

$$A_b(r) := \{x \in X | F(x, b) \leq r\}$$

Assume that r is a real number that $A_b(r) \neq \emptyset$. It is clear that $A_b(r)$ is a b-closed convex subset of X.

Theorem 2.3. *Let $(X, \langle ., .|.\rangle)$ be a 2-inner product space over the real field $\mathbb{R}$ and let $F : X\times < b > \longrightarrow \mathbb{R}$ be a convex 2-functional on $X\times < b >$. Let $x_0 \in X \setminus (A_b(r)+ < b >)$ and $g_0 \in A_b(r)$. Then the following are equivalent.*
(1) $g_0 = P_{A_b(r),b}(x_0)$;
(2) $F(x, b) \geq r + \frac{F(x_0,b) - r}{\|x_0 - g_0, b\|^2}\langle x - g_0, x_0 - g_0|b\rangle$ *for all* $x \in A_b(r)$ *where* $r = F(g_0, b)$.

Proof. (1) ⇒ (2) Assume that $g_0 = P_{A_b(r),b}(x_0)$. Since $x_0 \in X \setminus (A_b(r)+ < b >)$ we have $F(x_0, b) > r$. Let $x \in A_b(r)$, then $F(x, b) \leq r$. Set $\alpha = F(x_0, b) - r$, $\beta = r - F(x, b)$. Then $\alpha > 0$, $\beta \geq 0$ and $0 < \alpha + \beta = F(x_0, b) - F(x, b)$. Consider the element $u = \frac{\alpha x + \beta x_0}{\alpha + \beta}$. By convexity of F, for $y = y' = b, \lambda = \frac{\alpha}{\alpha + \beta}, \mu = 1$ and $a = b = 1$ in definition of convex 2-functional we have that:

$$\begin{aligned} F(u, b) = F(\frac{\alpha x + \beta x_0}{\alpha + \beta}, b) &\leq \frac{\alpha F(x, b) + \beta F(x_0, b)}{\alpha + \beta} \\ &= \frac{(F(x_0, b) - r)F(x, b) + (r - F(x, b))F(x_0, b)}{F(x_0, b) - F(x, b)} = r \end{aligned}$$

That is $u \in A_b(r)$. As $g_0 = P_{A_b(r),b}(x_0)$, we have by theorem (2.1) that $x_0 - g_0 \in (A_b(r) - g_0)^\circ_b$, so, $\langle g - g_0, x_0 - g_0|b\rangle \leq 0$ for all $g \in A_b(r)$. In particular, $\langle u - g_0, x_0 - g_0|b\rangle \leq 0$. That is

$$
\begin{aligned}
0 &\geq \langle u - g_0, x_0 - g_0|b\rangle = \langle \frac{\alpha x + \beta x_0}{\alpha + \beta} - g_0, x_0 - g_0|b\rangle \\
&= \frac{1}{\alpha+\beta}\langle \alpha x + \beta x_0 - (\alpha+\beta)g_0, x_0 - g_0|b\rangle \\
&= \frac{\alpha}{\alpha+\beta}\langle x - g_0, x_0 - g_0|b\rangle + \frac{\beta}{\alpha+\beta}\langle x_0 - g_0, x_0 - g_0|b\rangle \\
&= \frac{(F(x_0,b) - r)}{F(x_0,b) - F(x,b)}\langle x - g_0, x_0 - g_0|b\rangle + \frac{(r - F(x,b))}{F(x_0,b) - F(x,b)}\|x_0 - g_0, b\|^2
\end{aligned}
$$

Thus

$$
F(x,b) \geq \frac{(F(x_0,b) - r)}{\|x_0 - g_0, b\|^2}\langle x - g_0, x_0 - g_0|b\rangle
$$

for all $x \in A_b(r)$. Since above theorem is true for all $x \in A_b(r)$ so for $x = g_0$ we have that $F(g_0, b) \geq r$. But since $g_0 \in A_b(r)$ we have $F(g_0,b) \leq r$. Thus $F(g_0,b) = r$.
$(2) \Longrightarrow (1)$: Assume that (2) holds. Then for all $x \in A_b(r)$,

$$
0 \geq F(x,b) - r \geq \frac{(F(x_0,b) - r)}{\|x_0 - g_0, b\|^2}\langle x - g_0, x_0 - g_0|b\rangle
$$

Since $F(x_0,b) - y > 0$, we have that

$$
\langle x - g_0, x_0 - g_0|b\rangle \leq 0
$$

for all $x \in A_b(r)$. That is, $x_0 - g_0 \in (A_b(r) - g_0)^\circ$, whence, $g_0 = P_{A_b(r),b}(x_0)$. ☐

Corollary 2.5. *Let $f : X\times < b > \longrightarrow \mathbb{R}$ be a continues sublinear 2-functional on the 2-inner product space $(X, \langle ., .|.\rangle)$. Put $K_b(f) := \{x \in X | f(x,b) \leq 0\}$. Let $x_0 \in X \setminus (K_b(f) + < b >)$ and $g_0 \in K_b(f)$. Then the following statements are equivalent:*
(1) $g_0 = P_{K_b(f),b}(x_0)$;
(2) $f(x,b) \geq \frac{f(x_0,b)}{\|x_0 - g_0, b\|^2}\langle x - g_0, x_0 - g_0|b\rangle$ for all $x \in K_b(f)$.

Proof. It is sufficient that in above theorem taking $F = f$ and $r = 0$. ☐

It is clear that $X = K_b(f) \cup K_b(-f)$ and $\ker(f) = \{x \in X | f(x,b) = 0\} = K_b(f) \cap K_b(-f)$. If in the above corollary replacing f with $-f$, then we have:

Corollary 2.6. *Let $f : X\times < b > \longrightarrow \mathbb{R}$ be a continuous sublinear 2-functional on the 2-inner product space $(X, \langle ., .|.\rangle)$. Let $x_0 \in X \setminus (\ker(f) + < b >)$ and $g_0 \in \ker(f)$. Then the following statements are equivalent:*
(1) $g_0 = P_{\ker(f),b}(x_0)$;
(2) $f(x,b) = \frac{f(x_0,b)}{\|x_0 - g_0, b\|^2}\langle x, x_0 - g_0|b\rangle$ for all $x \in \ker(f)$.

Remark 2.7. For another proof of above corollary see [15].

3. B-METRIC PROJECTION IN 2-INNER PRODUCT SPACES

In this section we investigate some properties of the b-metric projection onto convex cone and get some consequence, specially we show that every 2-inner product space is direct sum of any b-chebyshev subspace and its b-orthogonal complement.

Proposition 3.1. *Let K be a convex b-chebyshev set and $K\cap <b>=\emptyset$. Then*
(1) $P_{K,b}$ is idempotent i.e.

$$P_{K,b}(P_{K,b}(x)) = P_{K,b}(x)$$

for every $x \in X$.
(2) $P_{K,b}$ is firmly nonexpansive i.e.

$$\langle x-y, P_{K,b}(x) - P_{K,b}(y)|b\rangle \geq \|P_{K,b}(x) - P_{K,b}(y), b\|^2$$

for all $x, y \in X$.
(3) $P_{K,b}$ is monotone i.e.

$$\langle x-y, P_{K,b}(x) - P_{K,b}(y)|b\rangle \geq 0$$

for all $x, y \in X$.
(4) $P_{K,b}$ is strictly nonexpansive i.e.

$$\|x-y,b\|^2 > \|P_{K,b}(x) - P_{K,b}(y), b\|^2 + \|x - P_{K,b}(x) - (y - P_{K,b}(y)), b\|^2$$

for all $x, y \in X$.
(5) $P_{K,b}$ is nonexpansive i.e.

$$\|P_{K,b}(x) - P_{K,b}(y), b\| \leq \|x-y,b\|$$

for all $x, y \in X$.
(6) $P_{K,b}$ is uniformly continuous.

Proof. (1) It is clear.
(2) We have:

$$\begin{aligned}\langle x-y, P_{K,b}(x) - P_{K,b}(y)|b\rangle &= \langle x - P_{K,b}(x), P_{K,b}(x) - P_{K,b}(y)|b\rangle \\ &+ \langle P_{K,b}(x) - P_{K,b}(y), P_{K,b}(x) - P_{K,b}(y)|b\rangle \\ &+ \langle P_{K,b}(y) - y, P_{K,b}(x) - P_{K,b}(y)|b\rangle\end{aligned}$$

The first and third terms on the right are nonnegative by theorem (2.1), and the second term is $\|P_{K,b}(x) - P_{K,b}(y), b\|^2$. This verifies (2).
(3) It is immediate consequence of (2).
(4) Using (2) we obtain for each $x, y \in X$ that:

$$\begin{aligned}\|x-y,b\|^2 &= \|(x - P_{K,b}(x)) + (P_{K,b}(x) - P_{K,b}(y)) + (P_{K,b}(y) - y), b\|^2 \\ &= \|P_{K,b}(x) - P_{K,b}(y), b\|^2 + \|(x - P_{K,b}(x)) - (y - P_{K,b}(y))\|^2 \\ &+ 2\langle P_{K,b}(x) - P_{K,b}(y), x - P_{K,b}(x) - (y - P_{K,b}(y))|b\rangle \\ &= \|P_{K,b}(x) - P_{K,b}(y), b\|^2 + \|(x - P_{K,b}(x)) - (y - P_{K,b}(y))\|^2 \\ &+ 2\langle P_{K,b}(x) - P_{K,b}(y), x - y|b\rangle - 2\|P_{K,b}(x) - P_{K,b}(y), b\|^2 \\ &\geq \|P_{K,b}(x) - P_{K,b}(y), b\|^2 + \|x - P_{K,b}(x) - (y - P_{K,b}(y)), b\|^2\end{aligned}$$

This proves (4).
(5) It is follows immediately from (4).
(6) It is follows immediately from (5). □

Theorem 3.1. *Let C be a b-chebyshev convex cone in the 2-inner product space X and $C\cap <b>=\emptyset$. Then C_b° is a b-chebyshev convex cone and*
(1) For each $x \in X$,

$$x = P_{C,b}(x) + P_{C_b^\circ,b}(x) \text{ and } P_{C,b}(x) \perp_b P_{C_b^\circ,b}(x).$$

Moreover, this representation is unique in the sense that if $x = y + z$ for some $y \in C$ and $z \in C_b^\circ$ with $y \perp_b z$, then $y = P_{C,b}(x)$ and $z = P_{C_b^\circ,b}(x)$.

(2) $\|x,b\|^2 = \|P_{C,b}(x),b\|^2 + \|P_{C_b^\circ,b}(x),b\|^2$ *for all* $x \in X\setminus <b>$.
(3) $C_b^\circ = \{x \in X | P_{C,b}(x) = 0\}$ *and* $C = \{x \in X | P_{C_b^\circ,b}(x) = 0\} = \{x \in X | P_{C,b}(x) = x\}$.
(4) $\|P_{C,b}(x),b\| \le \|x,b\|$ *for all* $x \in X$*; moreover,* $\|P_{C,b}(x),b\| = \|x,b\|$ *if and only if* $x \in C$.
(5) $C_b^{\circ\circ} = C$.
(6) $P_{C,b}$ *is positively homogeneous. i.e.*

$$P_{C,b}(\lambda x) = \lambda P_{C,b}(x) \text{ for all } x \in X, \lambda \ge 0.$$

Proof. (1) Let $x \in X$ and $c_0 = x - P_{C,b}(x)$. By theorem (2.3) $c_0 \in C_b^\circ$ and $c_0 \perp_b (x - c_0)$. For every $y \in C_b^\circ$,

$$\langle x - c_0, y|b\rangle = \langle P_{C,b}(x), y|b\rangle \le 0.$$

Hence $x - c_0 \in (C_b^\circ)^\circ$. By theorem (2.3), we get that $c_0 = P_{C_b^\circ,b}(x)$. This proves that C_b° is b-chebyshev convex cone, $x = P_{C,b}(x) + P_{C_b^\circ,b}(x)$ and $P_{C,b}(x) \perp_b P_{C_b^\circ,b}(x)$. Now we verify the uniqueness of this representation. Let $x = y + z$, where $y \in C, z \in C_b^\circ$ and $y \perp_b z$. For each $c \in C$,

$$\langle x - y, c|b\rangle = \langle z, c|b\rangle \le 0$$

and

$$\langle x - y, y|b\rangle = \langle z, y|b\rangle = 0.$$

By theorem (2.3) $y = P_{C,b}(x)$. Similarly $z = P_{C_b^\circ,b}(x)$.
(2) It is clear by (1) and Pythagorean theorem in 2-inner product spaces.
(3) By using (1) we have that:

$$x \in C_b^\circ \iff x = P_{C_b^\circ,b}(x) \iff P_{C,b}(x) = 0,$$

and

$$x \in C \iff x = P_{C,b}(x) \iff P_{C_b^\circ,b}(x) = 0.$$

(4) From (2), it is clear that $\|P_{C,b}(x),b\| \le \|x,b\|$ for all $x \in X$. Also from (2), $\|P_{C,b}(x),b\| = \|x,b\|$ if and only if $P_{C_b^\circ,b}(x) = 0$, which from (3) is equivalent to $x \in C$.
(5) By (3) we have:

$$C_b^{\circ\circ} = (C_b^\circ)^\circ = \{x \in X | P_{C_b^\circ,b}(x) = 0\} = C.$$

(6) Let $x \in X$ and $\lambda \ge 0$. Then $x = P_{C,b}(x) + P_{C_b^\circ,b}(x)$ and $\lambda x = \lambda P_{C,b}(x) + \lambda P_{C_b^\circ,b}(x)$. Since both C and C_b° are convex cones, $\lambda P_{C,b}(x) \in C$ and $\lambda P_{C_b^\circ,b}(x) \in C_b^\circ$. By (1) and the uniqueness of representation for λx, we see that $P_{C,b}(\lambda x) = \lambda P_{C,b}(x)$. □

Corollary 3.2. *Let* M *be a b-chebyshev subspace of the 2-inner product space* X*. Then* $M_b^\perp$ *is a b-chebyshev subspace and:*
(1) $x = P_{M,b}(x) + P_{M_b^\perp,b}(x)$*, for each* $x \in X$*. Moreover, this representation is unique in the sense that if* $x = y + z$ *where* $y \in M$ *and* $z \in M_b^\perp$*, then* $y = P_{M,b}(x)$ *and* $z = P_{M_b^\perp,b}(x)$.
(2) $\|x,b\|^2 = \|P_{M,b}(x),b\|^2 + \|P_{M_b^\perp,b}(x),b\|^2$ *for all* $x \in X\setminus <b>$.
(3) $M_b^\perp = \{x \in X | P_{M,b}(x) = 0\}$ *and* $M = \{x \in X | P_{M_b^\perp,b}(x) = 0\} = \{x \in X | P_{M,b}(x) = x\}$.
(4) $\|P_{M,b}(x),b\| \le \|x,b\|$ *for all* $x \in X$*; moreover,* $\|P_{M,b}(x),b\| = \|x,b\|$ *if and only if* $x \in M$.
(5) $M_b^{\perp\perp} = M$.

Theorem 3.2. *Let M be a b-chebyshev subspace of the 2-inner product space X and $M\cap <b>=\emptyset$. Then:*
(1) $P_{M,b}$ is a bounded linear 2-functional and $\|P_{M,b}\| = 1$ (in the case $M = \{0\}$, we have $\|P_{M,b}\| = 0$).
(2) $P_{M,b}$ is self-adjoint i.e.

$$\langle P_{M,b}(x), y|b\rangle = \langle x, P_{M,b}(y)|b\rangle \text{ for all } x, y \in X.$$

(3) For every $x \in X$,

$$\langle P_{M,b}(x), x|b\rangle = \|P_{M,b}(x), b\|^2.$$

(4) $P_{M,b}$ is nonnegative i.e.

$$\langle P_{M,b}(x), x|b\rangle \geq 0 \text{ for every } x \in X.$$

Proof. (1) Let $x, y \in X$ and $\alpha, \beta \in \mathbb{R}$. By remark (2.4), $x - P_{M,b}(x)$ and $y - P_{M,b}(y)$ are in $M_b^{\perp}$. since $M_b^{\perp}$ is subspace,

$$\alpha x + \beta y - (\alpha P_{M,b}(x) + \beta P_{M,b}(y)) = \alpha(x - P_{M,b}(x)) + \beta(y - P_{M,b}(y)) \in M_b^{\perp}.$$

Since $\alpha P_{M,b}(x) + \beta P_{M,b}(y) \in M$, remark (2.4) implies that $\alpha P_{M,b}(x) + \beta P_{M,b}(y) = P_{M,b}(\alpha x+\beta y)$. Thus $P_{M,b}$ is linear.From corollary (3.3 [4]) we get $\|P_{M,b}(x)\| \leq \|x, b\|$ for all $x \in X$. So $\|P_{M,b}$ is bounded and $\|P_{M,b}\| \leq 1$. Since $P_{M,b}y = y$ for all $y \in M$ and $\|y, b\| = \|P_{M,b}(y)\| \leq \|P_{M,b}\|\|y, b\|$, implies that $\|P_{M,b}\| \geq 1$ therefore $\|P_{M,b}\| = 1$.
(2) By remark (2.4), for each $x, y \in X$ we have $\langle P_{M,b}(x), y - P_{M,b}(y)|b\rangle = 0$, and hence

$$\langle P_{M,b}(x), y|b\rangle = \langle P_{M,b}(x), P_{M,b}(y)|b\rangle. \ (*)$$

By replacing x and y in above relation we obtain

$$\begin{aligned}\langle x, P_{M,b}(y)|b\rangle &= \langle P_{M,b}(y), x|b\rangle \\ &= \langle P_{M,b}(y), P_{M,b}(x)|b\rangle \\ &= \langle P_{M,b}(x), P_{M,b}(y)|b\rangle \\ &= \langle P_{M,b}(x), y|b\rangle.\end{aligned}$$

(3) Tacking $y = x$ in $(*)$.
(4) It is clear from (3). □

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