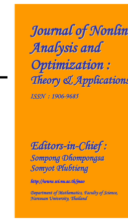
Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 4, No. 2, (2013), 149-155
ISSN : 1906-9605
http://www.math.sci.nu.ac.th



BANACH FIXED POINT THEOREM IN APPLICATION TO NONLINEAR ELLIPTIC SYSTEMS

G.A. AFROUZI[1,*], QIHU ZHANG[2] AND Z. NAGHIZADEH[1]

[1]Department of Mathematics, Faculty of Mathematical Sciences,
University of Mazandaran, Babolsar, Iran
[2]Department of Mathematics and Information Science,
Zhengzhou University of Light Industry,Zhengzhou, Henan 450002, China

ABSTRACT. This paper is concerned with some nonlinear elliptic systems. Under suitable conditions on the nonlinearities f and g, we obtain weak solution in Sobolev space $H = H_0^1(\Omega) \times H_0^1(\Omega)$ by applying the Banach fixed point theorem.

KEYWORDS: Weak solution; Nonlinear elliptic system; The Laplace operator; Banach fixed point theorem.
AMS Subject Classification: 35B30 35J60 35P15

1. INTRODUCTION

We study the nonlinear elliptic system of the form:

$$\begin{cases} -\Delta u - \operatorname{div}(h_1(|\nabla u|^2)\nabla u) = \lambda f(x,u,v) & \text{in } \Omega \\ -\Delta v - \operatorname{div}(h_2(|\nabla v|^2)\nabla v) = \lambda g(x,u,v) & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth open set in $\mathbb{R}^N$, $(N \geq 3)$, λ is a real number, $-\Delta u = \operatorname{div}(\nabla u)$ is the Laplacian of u, and $h_1, h_2 \in C(\mathbb{R}, \mathbb{R})$.

Theorems concerning the existence and properties of fixed point are known as fixed-point theorems. Such theorems are the most important tools for proving the existence and uniqueness of the solutions to various mathematical models(differential, integral and partial differential equations, and variational inequalities, etc.) representing phenomena arising in different fields, such as steady-state temperature distribution, chemical reactions, neutron transport theory, economic theories, epidemics and flow of fluids. They are also used to study the problems of optimal control related to this systems. For details, one can refer to [1, 3].

* *Corresponding author.*
Email addresses : afrouzi@umz.ac.ir (G.A. Afrouzi), zhangqh1999@yahoo.com.cn (Qihu Zhang), z.naghizadeh@umz.ac.ir (Z. Naghizadeh).
Article history : Received 30 May 2012. Accepted 13 June 2013.

In recent years, many publications have appeared concerning quasilinear elliptic systems which have been used in a great variety of applications, we refer the readers to [5, 6, 7, 8, 9] and the references therein. J. Zhang and Z. Zhang [11] used variational methods to obtain weak solution of semilinear elliptic system and quasilinear elliptic system.

Motivated by [4], in this paper, we will discuss problem (1.1). Under the suitable condition on the nonlinearities $f(x, u, v)$ and $g(x, u, v)$, using Banach fixed point theorem (see [10]), we show that system (1.1) has a unique weak solutions.

Throughout this paper for $(u, v) \in \mathbb{R}^2$, denote $|(u, v)|^2 = |u|^2 + |v|^2$. We assume that f and g are L^2 functions which are Lipschitz continuous with respect to the second variable, i.e., there exist constants $c_1, c_2 > 0$ such that for a.e. $x \in \Omega$ and for any $(u, v), (u_1, v_1) \in \mathbb{R}^2$

$$|f(x, u, v) - f(x, u_1, v_1)| \leq c_1|(u, v) - (u_1, v_1)| \tag{1.2}$$

$$|g(x, u, v) - g(x, u_1, v_1)| \leq c_2|(u, v) - (u_1, v_1)|. \tag{1.3}$$

We assume that h_1 and h_2 are the continuous and nondecreasing functions satisfying the following growth conditions:
There exist β_1, β_2 and $M_1, M_2 > 0$ such that

$$0 < h_1(t) \leq \beta_1 \; 0 < h_2(t) \leq \beta_2. \tag{1.4}$$

and

$$|h_1'(t)| \leq \frac{M_1}{1+t} \; |h_2'(t)| \leq \frac{M_2}{1+t} \tag{1.5}$$

for all $t \in R$.

Let λ_1 be the first eigenvalue of Dirichlet problem $-\Delta u = \lambda u$. The main result of this paper is as follows:

Definition 1.1. We say that $(u, v) \in H$ is a weak solution of (1.1) if

$$\int_\Omega [\nabla u \nabla \xi + \nabla v \nabla \eta + h_1(|\nabla u|^2)\nabla u \nabla \xi + h_2(|\nabla v|^2)\nabla v \nabla \eta] \, dx$$
$$-\lambda \int_\Omega [f(x, u, v)\xi + g(x, u, v)\eta] \, dx = 0$$

for all $(\xi, \eta) \in H$

Theorem 1.2. *Suppose that conditions* $(2) - (5)$ *hold. For any* $\lambda \in (0, \frac{\lambda_1}{c_1+c_2})$ *there exists a unique weak solution of (1.1).*

This paper is organized as follows. In section 2, we present some relevant lemmas. We reserve the section 3 for the proof of the main result.

2. Preliminary lemmas

Given a bounded smooth open set $\Omega \subset \mathbb{R}^N$. Let us consider the Hilbert space $H = H_0^1(\Omega) \times H_0^1(\Omega)$ and $\langle \, , \, \rangle_{L^2}$ the inner product in $L^2(\Omega)$. The norm on H is given by

$$\|(u, v)\| = (\int_\Omega (|\nabla u|^2 + |\nabla v|^2) \, dx)^{\frac{1}{2}}$$

and the norm on $L^2(\Omega) \times L^2(\Omega)$ is given by

$$\|(u, v)\|_{L^2 \times L^2} = (\int_\Omega (|u|^2 + |v|^2) \, dx)^{\frac{1}{2}}.$$

First, we define the operator $a : H \times H \to \mathbb{R}$ by

$$a((u, v), (\xi, \eta)) \quad = \quad \int_\Omega \nabla u \nabla \xi dx + \int_\Omega \nabla v \nabla \eta dx + \int_\Omega h_1(|\nabla u|^2)\nabla u \nabla \xi dx$$

$$+\int_\Omega h_2(|\nabla v|^2)\nabla v\nabla\eta dx,$$

respectively $b_\lambda : H\times H\to\mathbb{R}$ by

$$b_\lambda((u,v),(\xi,\eta))=\lambda[\int_\Omega f(x,u,v)\xi dx+\int_\Omega g(x,u,v)\eta dx].$$

We point out certain properties of the operators a, respectively b_λ.

Lemma 2.1. The operators a and b_λ satisfy the following properties:

$(A1)$ for each $(u,v)\in H$, the application $(\xi,\eta)\mapsto a((u,v),(\xi,\eta))$ is linear and continuous.

$(A2)$ $a((u,v),\ (u,v)-(u_1,v_1))-a((u_1,v_1),\ (u,v)-(u_1,v_1))\geq\|(u,v)-(u_1,v_1)\|^2$
for all $(u,v),(u_1,v_1)\in H$.

$(A3)$ there exist $M>0$ such that

$$|\ a((u,v),(\xi,\eta))-a((u_1,v_1),(\xi,\eta))|\leq M\|(u,v)-(u_1,v_1)\|\ \|(\xi,\eta)\|$$

for all $(u,v),(u_1,v_1),(\xi,\eta)\in H$.

$(B1)$ for each $(u,v)\in H$, the application $(\xi,\eta)\mapsto b_\lambda((u,v),(\xi,\eta))$ is linear and continuous.

$(B2)$ $b_\lambda((u,v),\ (u,v)-(u_1,v_1))-b_\lambda((u_1,v_1),\ (u,v)-(u_1,v_1))\leq\dfrac{\lambda(c_1+c_2)}{\lambda_1}\ \|(u,v)-(u_1,v_1)\|^2$
for all $(u,v),(u_1,v_1)\in H$.

$(B3)$ there exist $N=N(\lambda)>0$ such that

$$|\ b_\lambda((u,v),(\xi,\eta))-b_\lambda((u_1,v_1),(\xi,\eta))|\leq N\|(u,v)-(u_1,v_1)\|\ \|(\xi,\eta)\|$$

for all $(u,v),(u_1,v_1),(\xi,\eta)\in H$.

Proof. $(A1)$ We fix $(u,v)\in H$. It is clear that $(\xi,\eta)\mapsto a((u,v),\ (\xi,\eta))$ is linear. On the other hand, using Holder's inequality we have

$$\begin{aligned}|a((u,v),(\xi,\eta))| &= |\int_\Omega\nabla u\nabla\xi dx+\int_\Omega\nabla v\nabla\eta dx+\int_\Omega h_1(|\nabla u|^2)\nabla u\nabla\xi dx\\ &\quad+\int_\Omega h_2(|\nabla v|^2)\nabla v\nabla\eta dx|\\ &\leq (\beta_1+1)\|(u,v)\|\ \|(\xi,\eta)\|+(\beta_2+1)\|(u,v)\|\ \|(\xi,\eta)\|\\ &= (\beta_1+\beta_2+2)\|(u,v)\|\ \|(\xi,\eta)\|.\end{aligned}$$

It follows that $(\xi,\eta)\mapsto a((u,v),(\xi,\eta))$ is continuous.
$(A2)$ we have

$$\begin{aligned}&a((u,v),\ (u,v)-(u_1,v_1))-a((u_1,v_1),\ (u,v)-(u_1,v_1))\\ &=\|(u,v)-(u_1,v_1)\|^2+[\int_\Omega(h_1(|\nabla u|^2)|\nabla u|^2+h_1(|\nabla u_1|^2)|\nabla u_1|^2-h_1(|\nabla u|^2)\nabla u\nabla u_1\\ &-h_1(|\nabla u_1|^2)\nabla u\nabla u_1)\ dx]+[\int_\Omega(h_2(|\nabla v|^2)|\nabla v|^2+h_2(|\nabla v_1|^2)|\nabla v_1|^2-h_2(|\nabla v|^2)\nabla v\nabla v_1\\ &-h_2(|\nabla v_1|^2)\nabla v\nabla v_1)\ dx]\end{aligned}$$

on the other hand we have

$$h_1(|\nabla u|^2)|\nabla u|^2+h_1(|\nabla u_1|^2)|\nabla u_1|^2\geq h_1(|\nabla u|^2)\nabla u\nabla u_1+h_1(|\nabla u_1|^2)\nabla u\nabla u_1. \tag{2.1}$$

Indeed, it is sufficient to show that

$$h_1(|\nabla u|^2)|\nabla u|^2+h_1(|\nabla u_1|^2)|\nabla u_1|^2\geq|h_1(|\nabla u|^2)\nabla u\nabla u_1+h_1(|\nabla u_1|^2)\nabla u\nabla u_1|$$

or

$$h_1(|\nabla u|^2)|\nabla u|^2+h_1(|\nabla u_1|^2)|\nabla u_1|^2\geq h_1(|\nabla u|^2)|\nabla u.\nabla u_1|+h_1(|\nabla u_1|^2)|\nabla u.\nabla u_1|.$$

So we shall prove that

$$h_1(|\nabla u|^2)|\nabla u|^2 + h_1(|\nabla u_1|^2)|\nabla u_1|^2 \geq h_1(|\nabla u|^2)|\nabla u||\nabla u_1| + h_1(|\nabla u_1|^2)|\nabla u||\nabla u_1|$$

or

$$(h_1(|\nabla u|^2)|\nabla u| - h_1(|\nabla u_1|^2)|\nabla u_1|)\ (|\nabla u| - |\nabla u_1|) \geq 0 \tag{2.2}$$

we define the auxiliary function $\psi_1 : [0,\infty) \to R$ by $\psi_1(t) = h_1(t^2)t$. Obviously ψ_1 is increasing on $[0,\infty)$ which implies

$$[\psi(t_1) - \psi(t_2)]\ (t_1 - t_2) \geq 0\ \forall t_1, t_2 \in [0,\infty).$$

Taking $t_1 = |\nabla u|, t_2 = |\nabla u_1|$ the inequality (7) follows.

Similarly by auxiliary function $\psi_2 : [0,\infty) \to R$ by $\psi_2(t) = h_2(t^2)t$ we obtain

$$h_2(|\nabla v|^2)|\nabla v|^2 + h_2(|\nabla v_1|^2)|\nabla v_1|^2 - h_2(|\nabla v|^2)\nabla v\nabla v_1 - h_2(|\nabla v_1|^2)\nabla v\nabla v_1 \geq 0.$$

Hence

$$a((u,v),\ (u,v)-(u_1,v_1)) - a((u_1,v_1),\ (u,v)-(u_1,v_1)) \\ \geq \|(u,v)-(u_1,v_1)\|^2,\ \forall (u,v),(u_1,v_1) \in H.$$

$(A3)$

$$|\ a((u,v),(\xi,\eta)) - a((u_1,v_1),(\xi,\eta))| = |\int_\Omega (\nabla u - \nabla u_1)\nabla\xi dx + \int_\Omega (\nabla v - \nabla v_1)\nabla\eta dx$$
$$+ \int_\Omega [h_1(|\nabla u|^2)\nabla u - h_1(|\nabla u_1|^2)\nabla u_1]\nabla\xi dx + \int_\Omega [h_2(|\nabla v|^2)\nabla v - h_2(|\nabla v_1|^2)\nabla v_1]\nabla\eta dx|$$
$$\leq 2\|(u,v)-(u_1,v_1)\|\ \|(\xi,\eta)\| + \int_\Omega \Sigma_{j=1}^N\ |h_1(|\nabla u|^2)\frac{\partial u}{\partial x_j} - h_1(|\nabla u_1|^2)\frac{\partial u_1}{\partial x_j}|\ |\frac{\partial \xi}{\partial x_j}|$$
$$+ \int_\Omega \Sigma_{j=1}^N\ |h_2(|\nabla v|^2)\frac{\partial v}{\partial x_j} - h_2(|\nabla v_1|^2)\frac{\partial v_1}{\partial x_j}|\ |\frac{\partial \eta}{\partial x_j}|$$

on the other hand we have

$$|h_1(|\nabla u|^2)\frac{\partial u}{\partial x_j} - h_1(|\nabla u_1|^2)\frac{\partial u_1}{\partial x_j}| \leq (\beta_1 + M_1(N+1))|\nabla u - \nabla u_1|$$

and

$$|h_2(|\nabla v|^2)\frac{\partial v}{\partial x_j} - h_2(|\nabla v_1|^2)\frac{\partial v_1}{\partial x_j}| \leq (\beta_2 + M_2(N+1))|\nabla v - \nabla v_1|.$$

Indeed let $x \in \overline{\Omega}$ be fixed and $H_i : R^N \to R$ be defined by

$$H_i(\xi) = h_1(|\xi|^2)\xi_i\ \forall \xi \in R^N,\ \forall i \in \{1,2,...N\}.$$

Using the mean value theorem we deduce that

$$|H_i(\xi) - H_i(\eta)| \leq |\xi - \eta| \sup_{\theta \in [\xi,\eta]} |\nabla H_i(\theta)|$$

where $[\xi,\eta]$ is the line segment in R^N between the point ξ and η, i.e., $[\xi,\eta] = \{t\xi + (1-t)\eta :\ t \in [0,1]\ \}$. But

$$|\nabla H_i(\theta)| = (\Sigma_{j=1}^N\ (\frac{\partial H_i(\theta)}{\partial \theta_j})^2)^{\frac{1}{2}} \leq \Sigma_{j=1}^N\ |\frac{\partial H_i(\theta)}{\partial \theta_j}|. \tag{2.3}$$

For $j \neq i$

$$\frac{\partial H_i(\theta)}{\partial \theta_j} = 2\ h_1'(|\theta|^2)\ \theta_i\ \theta_j$$

and for $j = i$

$$\frac{\partial H_i(\theta)}{\partial \theta_i} = h_1(|\theta|^2) + 2\ h_1'(|\theta|^2)\ \theta_i^2.$$

Thus by (4), (5), (8), we find

$$|\nabla H_i(\theta)| \leq \Sigma_{j=1}^N\ |\frac{\partial H_i(\theta)}{\partial \theta_j}| \leq h_1(|\theta|^2) + 2\ |h_1'(|\theta|^2)|\ \Sigma_{j=1}^N|\theta_i\theta_j|$$
$$\leq h_1(|\theta|^2) + 2\ |h_1'(|\theta|^2)|\Sigma_{j=1}^N\ \frac{\theta_i^2 + \theta_j^2}{2}$$

$$\leq \beta_1 + 2|h_1'(|\theta|^2)|\frac{N+1}{2}|\theta|^2$$
$$\leq \beta_1 + M_1(N+1).$$

Similarly for $G_i(\xi) = h_2(|\xi|^2)\xi_i$ we have $|\nabla G_i(\theta)| \leq \beta_2 + M_2(N+1)$.
Hence

$$|a((u,v),(\xi,\eta)) - a((u_1,v_1),\ (\xi,\eta))| \leq 2\|(u,v)-(u_1,v_1\|\ \|(\xi,\eta)\|$$
$$+(\beta_1 + M_1(N+1))\int_\Omega \Sigma_{j=1}^N\ |\nabla u - \nabla u_1|\ |\frac{\partial \xi}{\partial x_j}|dx$$
$$+(\beta_2 + M_2(N+1))\int_\Omega \Sigma_{j=1}^N\ |\nabla v - \nabla v_1|\ |\frac{\partial \eta}{\partial x_j}|dx$$
$$\leq 2\|(u,v)-(u_1,v_1)\|\ \|(\xi,\eta)\| + (\beta_1 + M_1(N+1))N(\int_\Omega |\nabla u - \nabla u_1|^2 dx)^{\frac{1}{2}}\ (\int_\Omega |\nabla \xi|^2 dx)^{\frac{1}{2}}$$
$$+(\beta_2 + M_2(N+1))N(\int_\Omega |\nabla v - \nabla v_1|^2 dx)^{\frac{1}{2}}\ (\int_\Omega |\nabla \eta|^2 dx)^{\frac{1}{2}} \leq M\|(u,v)-(u_1,v_1)\|\ \|(\xi,\eta)\|$$

where $M = 2 + N(\beta_1 + \beta_2 + (N+1)(M_1 + M_2))$.

$(B1)$ We fix $(u,v) \in H$. Obviously, the application $(\xi,\eta) \mapsto b_\lambda((u,v),\ (\xi,\eta))$ is linear. Using Holder's inequality, we have

$$|b_\lambda((u,v),\ (\xi,\eta))| = |\lambda \textstyle\int_\Omega [f(x,u,v)\xi + g(x,u,v)\eta]dx|$$
$$\leq \lambda(\textstyle\int_\Omega |f(x,u,v)|^2 dx)^{\frac{1}{2}}(\int_\Omega |\xi|^2 dx)^{\frac{1}{2}} + \lambda(\int_\Omega |g(x,u,v)|^2 dx)^{\frac{1}{2}}(\int_\Omega |\eta|^2 dx)^{\frac{1}{2}}$$
$$\leq K\|(\xi,\eta))\|,$$

where K is positive constant.

$(B2)$ By (2), (3) we have

$$b_\lambda((u,v),\ (u,v)-(u_1,v_1)) - b_\lambda((u_1,v_1),\ (u,v)-(u_1,v_1))$$
$$= \lambda \textstyle\int_\Omega [f(x,u,v) - f(x,u_1,v_1)](u-u_1)dx + \lambda \int_\Omega [g(x,u,v) - g(x,u_1,v_1)](v-v_1)dx$$
$$\leq \lambda c_1(\textstyle\int_\Omega |u-u_1|^2 + |v-v_1|^2)dx)^{\frac{1}{2}}(\int_\Omega |u-u_1|^2 dx)^{\frac{1}{2}}$$
$$+\lambda c_2(\textstyle\int_\Omega |u-u_1|^2 + |v-v_1|^2)dx)^{\frac{1}{2}}(\int_\Omega |v-v_1|^2 dx)^{\frac{1}{2}}$$
$$\leq \lambda c_1\|(u,v)-(u_1,v_1)\|^2_{L^2\times L^2} + \lambda c_2\|(u,v)-(u_1,v_1)\|^2_{L^2\times L^2}$$
$$= \lambda(c_1+c_2)\|(u,v)-(u_1,v_1)\|^2_{L^2\times L^2}$$
$$\leq \tfrac{\lambda(c_1+c_2)}{\lambda_1}\|(u,v)-(u_1,v_1)\|^2.$$

$(B3)$ Using Holder's inequality and (2), (3) we obtain

$$|b_\lambda((u,v),(\xi,\eta)) - b_\lambda((u_1,v_1),(\xi,\eta))|$$
$$= |\lambda \textstyle\int_\Omega [f(x,u,v) - f(x,u_1,v_1)]\xi dx + \lambda \int_\Omega [g(x,u,v) - g(x,u_1,v_1)]\eta dx|$$
$$\leq \lambda c_1\|(u,v)-(u_1,v_1)\|_{L^2\times L^2}\ (\textstyle\int_\Omega |\xi|^2 dx)^{\frac{1}{2}} + \lambda c_2\|(u,v)-(u_1,v_1)\|_{L^2\times L^2}\ (\int_\Omega |\eta|^2 dx)^{\frac{1}{2}}$$
$$\leq \lambda(c_1+c_2)\|(u,v)-(u_1,v_1)\|_{L^2\times L^2}\|(\xi,\eta)\|_{L^2\times L^2}$$
$$\leq \tfrac{\lambda(c_1+c_2)}{\lambda_1}\|(u,v)-(u_1,v_1)\|\ \|(\xi,\eta)\|$$
$$= N\|(u,v)-(u_1,v_1)\|\ \|(\xi,\eta)\|,$$

where $N = \frac{\lambda(c_1+c_2)}{\lambda_1}$.

3. Proof of main theorem

In this section we give the proof of theorem 1.2.

Proof of theorem 1.2. Let $\lambda \in (0, \frac{\lambda_1}{c_1+c_2})$ be arbitrary but fixed. By Lemma 2.1(A1) and the Riesz theorem (see e.g. Brezis [2], Theorem V.5) we deduce that for each $(u,v) \in H$ there exists a unique element denote by $A(u,v) \in H$ such that

$$a((u,v),(\xi,\eta)) = \langle A(u,v),(\xi,\eta)\rangle.$$

Thus we can define the operator $A : H \to H$. Using Lemma 2.1(A2) it follows that

$$\langle A(u,v) - A(u_1,v_1),\ (u,v) - (u_1,v_1)\rangle \geq \|(u,v) - (u_1,v_1)\|^2 \tag{3.1}$$

for all $(u,v), (u_1,v_1) \in H$ i.e., A is strongly monotone.
Lemma 2.1(A3) yields

$$|\langle A(u,v), (\xi,\eta)\rangle - \langle A(u_1,v_1), (\xi,\eta)\rangle| \leq M\|(u,v) - (u_1,v_1)\|\ \|(\xi,\eta)\|$$

for all $(u,v), (u_1,v_1), (\xi,\eta) \in H$. Hence,

$$\|A(u,v) - A(u_1,v_1)\| = \sup_{\|(\xi,\eta)\|\leq 1} |\langle A(u_1,v_1), (\xi,\eta)\rangle| \leq M\|(u,v) - (u_1,v_1)\| \tag{3.2}$$

i.e., A is Lipschitz continuous.

By Lemma 2.1$(B1)$ and the Riesz theorem we deduce that for each $(u,v) \in H$ there exists a unique element $B_\lambda(u,v) \in H$ such that

$$b_\lambda((u,v),(\xi,\eta)) = \langle B_\lambda(u,v),\ (\xi,\eta)\rangle,\ \forall(\xi,\eta) \in H.$$

Thus, we can also define the operator $B_\lambda : H \to H$ which satisfies

$$\langle B_\lambda(u,v),\ (u,v) - (u_1,v_1)\rangle - \langle B_\lambda(u,v),\ (u,v) - (u_1,v_1)\rangle \leq \frac{\lambda(c_1+c_2)}{\lambda_1}\|(u,v) - (u_1,v_1)\|^2. \tag{3.3}$$

Using Lemma 2.1$(B3)$ we find

$$\begin{aligned}\|B_\lambda(u,v) - B_\lambda(u_1,v_1)\| &= \sup|\langle B_\lambda(u,v), (\xi,\eta)\rangle - \langle B_\lambda(u_1,v_1), (\xi,\eta)\rangle| \\ &= \sup_{\|w\|\leq 1} |b_\lambda((u,v),(\xi,\eta)) - b_\lambda((u_1,v_1),(\xi,\eta))| \leq N\|(u,v) - (u_1,v_1)\|\end{aligned} \tag{3.4}$$

we define the operator $T : H \to H$ by

$$T(u,v) = (u,v) - t(A(u,v) - B_\lambda(u,v))$$

where $t \in (0,\ \frac{2(1-\frac{\lambda(c_1+c_2)}{\lambda_1})}{(N+M)^2}\ )$. The relation (9)-(12) shows that for each $(u,v), (u_1,v_1) \in H$ we have

$$\begin{aligned}\|T(u,v) - T(u_1,v_1)\|^2 &= \langle T(u,v) - T(u_1,v_1),\ T(u,v) - T(u_1,v_1)\rangle \\ &= \langle (u,v) - t(A(u,v) - B_\lambda(u,v)) - (u_1,v_1) + t(A(u_1,v_1) - B_\lambda(u_1,v_1)), \\ &\quad (u,v) - t(A(u,v) - B_\lambda(u,v)) - (u_1,v_1) + t(A(u_1,v_1) - B_\lambda(u_1,v_1))\rangle \\ &= \|(u,v) - (u_1,v_1)\|^2 - 2t\langle A(u,v) - A(u_1,v_1),\ (u,v) - (u_1,v_1)\rangle \\ &\quad +2t\langle B_\lambda(u,v) - B_\lambda(u_1,v_1),\ (u,v) - (u_1,v_1)\rangle - 2t^2\langle A(u,v) - A(u_1,v_1), \\ &\quad B_\lambda(u,v) - B_\lambda(u_1,v_1)\rangle + t^2\|A(u,v) - A(u_1,v_1)\|^2 + t^2\|B_\lambda(u,v) - B_\lambda(u_1,v_1)\|^2 \\ &\leq \|(u,v) - (u_1,v_1)\|^2 - 2t\|(u,v) - (u_1,v_1)\|^2 + 2t\tfrac{\lambda(c_1+c_2)}{\lambda_1}\|(u,v) - (u_1,v_1)\|^2 \\ &\quad +2t^2(M\|(u,v) - (u_1,v_1)\|)\ (N\|(u,v) - (u_1,v_1)\|) + t^2M^2\|(u,v) - (u_1,v_1)\|^2 \\ &\quad +t^2N^2\|(u,v) - (u_1,v_1)\|^2 \\ &= [1 - 2t(1 - \tfrac{\lambda(c_1+c_2)}{\lambda_1}) + M^2t^2 + N^2t^2 + 2NMt^2]\ \|(u,v) - (u_1,v_1)\|^2 \\ &= \beta\|(u,v) - (u_1,v_1)\|^2,\end{aligned}$$

where

$$\beta = 1 - 2(1 - \frac{\lambda(c_1+c_2)}{\lambda_1})t + (M+N)^2t^2 \geq 0.$$

If $t = 0$ or $t = \frac{2(1-\frac{\lambda(c_1+c_2)}{\lambda_1})}{(M+N)^2}$ then $\beta = 1$. This implies that $\sqrt{\beta} < 1$ for all $t \in (0,\ \frac{2(1-\frac{\lambda(c_1+c_2)}{\lambda_1})}{(M+N)^2})$.
Hence

$$\|T(u,v) - T(u_1,v_1)\| \leq \sqrt{\beta}\|(u,v) - (u_1,v_1)\|,\ \forall\ (u,v), (u_1,v_1) \in H$$

i,e., T is $\sqrt{\beta}$- contractive with $\sqrt{\beta} < 1$. By Banach fixed point theorem (see Zeidler [10], section 1.6) it follows that there is a unique solution $(u,v) \in H$ of problem $T(u,v) = (u,v)$ i.e., the problem $A(u,v) = B_\lambda(u,v)$ has a unique solution $(u,v) \in H$. It follows that

$$\langle A(u,v),\ (\xi,\eta)\rangle = \langle B_\lambda(u,v),\ (\xi,\eta)\rangle,\ \forall\ (\xi,\eta) \in H$$

$$a((u,v),(\xi,\eta)) = b_\lambda((u,v),(\xi,\eta)).$$

Thus the proof of Theorem 1.2 is complete.

Acknowledgments

The authors would like to thank the reviewers for their helpful comments.

References

1. H. Amann, *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Review 18 (1976), 620-709.
2. H. Brezis, *Analyse Fonctionnelle : theorie et applications, Masson, Paris, 1992.*
3. L. Collatz, *Functional analysis and numerical analysis,* Academic Press, New York, 1966.
4. N. Costea, M. Mihailescu, *On an eigenvalue problem involving variable exponent growth conditions,* Nonlinear Anal. 71 (2009), 4271-4278.
5. A. Djellit, S. Tas, *Existence of solutions for a class of elliptic systems in R^N involving the p-Laplacian,* Electronic J. Diff. Eqns. Vol. 2003(2003), No. 56, 1-8.
6. A. Djellit, S. Tas, *On some nonlinear elliptic systems,* Nonlinear Anal. 59 (2004), 695-706.
7. A. Djellit, S. Tas, *Quasilinear elliptic systems with critical Sobolev exponents in $\mathbb{R}^N$*, Nonlinear Anal. 66 (2007), 1485-1497.
8. P. Drabek, N. M. Stavrakakis, N. B. Zographopoulos, *Multiple nonsemitrivial solutions for quasilinear systems,* Differential Integral Equations 16 (12) (2003), 1519-1531.
9. D. D. Hai, H. Wang, *Nontrivial solutions for p-Laplacian systems,* J. Math. Anal. Appl 330. (2007), 186-194.
10. E. Zeidler, *Applied Functional Analysis: Applications in Mathematical Physics,* Springer-Verlag, New York, 1995.
11. J. Zhang, Z. Zhang, *Existence results for some nonlinear elliptic systems,* Nonlinear Anal. 71 (2009), 2840-2846.