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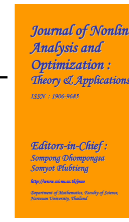

SOME SUBORDINATION RESULTS ASSOCIATED WITH GENERALIZED RUSCHEWEYH DERIVATIVES

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ABSTRACT. In this paper, we consider an unified class of functions of complex order associated with generalized Ruscheweyh derivative. We obtain a necessary and sufficient condition for functions to be in this class.

KEYWORDS : Starlike functions of complex order; Convex functions of complex order; Subordination; generalized Ruscheweyh derivative.
AMS Subject Classification: 30C45.

1. INTRODUCTION

Let A be the class of all analytic functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. Further, by S we shall denote the class of all functions in A which are univalent in U.

A function $f \in A$ is subordinate to an univalent function $g \in A$, written $f(z) \prec g(z)$, if $f(0) = g(0)$ and $f(U) \subseteq g(U)$. Let Ω be the family of analytic functions $w(z)$ in the unit disk U satisfying the conditions $w(0) = 0$, $|w(z)| < 1$ for $z \in U$. Note that $f(z) \prec g(z)$ if there is a function $w(z) \in \Omega$ such that $f(z) = g(w(z))$.

Let $\phi(z)$ be an analytic function with positive real part on U and $\phi(0) = 1$, $\phi'(0) > 0$ which maps the unit disc U onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Ma and Minda [3] introduced and studied the class $S^*(\phi)$, consists of functions $f \in S$ for which

$$\frac{zf'(z)}{f(z)} \prec \phi(z), \quad (z \in U).$$

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Recently, Ravichandran et al.[5] defined classes related to the class of starlike functions of complex order as follows:

Definition 1.1. Let $b \neq 0$ be a complex number. Let $\phi(z)$ be an analytic function with positive real part on U with $\phi(0) = 1, \phi'(0) > 0$ which maps the unit disk U onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Then the class $S_b^*(\phi)$ consists of all analytic functions $f \in A$ satisfying

$$1 + \frac{1}{b}(\frac{zf'(z)}{f(z)} - 1) \prec \phi(z).$$

The class $C_b(\phi)$ consists of functions $f \in A$ satisfying

$$1 + \frac{1}{b}\frac{zf''(z)}{f'(z)} \prec \phi(z).$$

Following the work of Ma and Minda [3], Shanmugam and Sivasubramanian [7] obtained Fekete-Szegö inequality for the more general class $M_\alpha(\phi)$, defined by

$$\frac{\alpha z^2 f''(z) + zf'(z)}{(1-\alpha)f(z) + \alpha z f'(z)} \prec \phi(z),$$

where $\phi(z)$ satisfies the conditions mentioned in Definition 1.1. Kamali et al.[2] introduced and studied a new class of functions $f \in T$ for which

$$Re(\frac{\alpha z^3 f'''(z) + (1+2\alpha)z^2 f''(z) + zf'(z)}{\alpha z^2 f''(z) + zf'(z)}) > \beta,\ 0 \leq \alpha < 1, 0 \leq \beta < 1.$$

Shanmugum et. al.[8] remarked that the class of functions T is the familiar class of functions introduced and studied by Silverman [10]. In a later investigation, this particular class introduced by Kamali and Akbulut was generalized by Shanmugum et al. [9]. Shanmugum et. al.[8] introduced a more general class of complex order $M[b, \alpha](\phi)$ defined as follows:

Definition 1.2. Let $b \neq 0$ be a complex number. Let $\phi(z)$ be an analytic function with positive real part on U with $\phi(0) = 1, \phi'(0) > 0$ which maps the unit disk U onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Then the class $M[b, \alpha](\phi)$ consists of all analytic functions $f \in A$ satisfying

$$1 + \frac{1}{b}(\frac{\alpha z^3 f'''(z) + (1+2\alpha)z^2 f''(z) + zf'(z)}{\alpha z^2 f''(z) + zf'(z)} - 1) \prec \phi(z),\ 0 \leq \alpha < 1.$$

Clearly,

$$M[b, 0](\phi) \equiv C_b(\phi).$$

where the class $C_b(\phi)$ consists of functions $f \in A$ satisfying

$$1 + \frac{1}{b}\frac{zf''(z)}{f'(z)} \prec \phi(z).$$

In the present paper we shall need a recent generalization of the Ruscheweyh derivative which was introduced in [1].

Let $f \in A$, $\lambda \geq 0$ and $m \in \mathbb{R}$, $m > -1$, then we consider

$$D_\lambda^m f(z) = \frac{z}{(1-z)^{m+1}} * D_\lambda f(z), \quad z \in U,$$

where $D_\lambda f(z) = (1-\lambda)f(z) + \lambda z f'(z), \quad z \in U.$

If $f \in A$, $f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in U$ we obtain the power series expansion of the form

$$D_\lambda^m f(z) = z + \sum_{n=2}^{\infty}[1 + (n-1)\lambda]\frac{(m+1)_{n-1}}{(1)_{n-1}} a_n z^n, \quad z \in U,$$

where $(a)_n$ is the Pochhammar symbol, given by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1 & \text{for } n = 0 \\ a(a+1)(a+2)...(a+n-1) & \text{for } n \in \mathbb{N} = \{1, 2, 3, ...\}. \end{cases}$$

In the case $m \in \mathbb{N} = \{1, 2, 3, ...\}$, we have

$$D_\lambda^m f(z) = \frac{z(z^{m-1}D_\lambda f(z))^{(m)}}{m!}, \quad z \in U,$$

and for $\lambda = 0$ we obtain the mth Ruscheweyh derivative introduced in [6], $D_0^m = D^m$,

Since

$$\frac{(m+1)_{n-1}}{(1)_{n-1}} = \frac{\Gamma(m+1+n-1)\Gamma(1)}{\Gamma(m+1)\Gamma(1+n-1)} = \frac{\Gamma(m+n)}{\Gamma(m+1)\Gamma(n)} = \frac{(m+n-1)!}{m!(n-1)!}.$$

And

$$\sigma(m,n) = \binom{m+n-1}{m} = \frac{(m+n-1)!}{m!(n-1)!}.$$

So, we get

$$\sigma(m,n) = \frac{(m+1)_{n-1}}{(1)_{n-1}}$$

Hence

$$D_\lambda^m f(z) = z + \sum_{n=2}^{\infty}[1+(n-1)\lambda]\sigma(m,n)a_n z^n, \quad z \in U,$$

So in this paper we introduce a more general class of complex order $H[b,\alpha,m,\lambda](\phi)$ which we define below.

Definition 1.3. Let $b \neq 0$ be a complex number. Let $\phi(z)$ be an analytic function with positive real part on U with $\phi(0) = 1$, $\phi'(0) > 0$ which maps the unit disc U onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Then the class $H[b,\alpha,m,\lambda](\phi)$ consists of all analytic functions $f \in A$ satisfying

$$1 + \frac{1}{b}\left(\frac{\alpha z^3(D_\lambda^m f)'''(z) + (1+2\alpha)z^2(D_\lambda^m f)''(z) + z(D_\lambda^m f)'(z)}{\alpha z^2(D_\lambda^m f)''(z) + z(D_\lambda^m f)'(z)} - 1\right) \prec \phi(z),$$
$$0 \le \alpha < 1, \lambda \ge o, m > -1.$$

Clearly,

$$H[b,\alpha,m,\lambda](\phi) \equiv M[b,\alpha](\phi)$$

where $m = \lambda = 0$, [8].

Motivated essentially by the aforementioned works, we obtain certain necessary and sufficient conditions for the unified class of functions $H[b,\alpha,m,\lambda](\phi)$ which we have defined. The motivation of this paper is to generalize the results obtained by Shanmugum et. al.[8].

2. PRELIMINARIES AND NOTATIONS

In order to prove our main results, we need the following lemmas.

Lemma 2.1. [5] *Let ϕ be a convex function defined on U, $\phi(0) = 1$. Define $F(z)$ by*

$$F(z) = z\, exp\left(\int_0^z \frac{\phi(x)-1}{x}dx\right). \tag{2.1}$$

Let $q(z) = 1 + c_1 z + ...$ be analytic in U. Then

$$1 + \frac{zq'(z)}{q(z)} \prec \phi(z), \tag{2.2}$$

if and only if for all $|s| \leq 1$ and $|t| \leq 1$, we have

$$\frac{q(tz)}{q(sz)} \prec \frac{sF(tz)}{tF(sz)}. \tag{2.3}$$

Lemma 2.2 ([4], Corrolary 3.4h.1,p.135)**.** *Let $q(z)$ be univalent in U and let $\varphi(z)$ be analytic in a domain containing $q(U)$. If $\frac{zq'(z)}{\varphi(q(z))}$ is starlike, then*

$$zp'(z)\varphi(p(z)) \prec zq'(z)\varphi(q(z)),$$

implies that $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

3. MAIN RESULTS

3.1. **Subordination Results.** Applying Lemma 2.1 we have

Theorem 3.1. *Let $\phi(z)$ and $F(z)$ be as in Lemma 2.1. The function $f \in H[b, \alpha, m, \lambda](\phi)$ if and only if for all $|s| \leq 1$ and $|t| \leq 1$, we have*

$$\left(\frac{s[\alpha z^2(D_\lambda^m f)''(tz) + z(D_\lambda^m f)'(tz)]}{t[\alpha z^2(D_\lambda^m f)''(sz) + z(D_\lambda^m f)'(sz)]}\right)^{\frac{1}{b}} \prec \frac{sF(tz)}{tF(sz)} \tag{3.1}$$

Proof. Define the function $p(z)$ by

$$p(z) = \left(\frac{\alpha z^2(D_\lambda^m f)''(z) + z(D_\lambda^m f)'(z)}{z}\right)^{\frac{1}{b}}. \tag{3.2}$$

By taking logarithmic derivative of $p(z)$ given by (3.2), we get

$$\frac{zp'(z)}{p(z)} = \frac{1}{b}\left\{\frac{\alpha z^3(D_\lambda^m f)'''(z) + (1+2\alpha)z^2(D_\lambda^m f)''(z) + z(D_\lambda^m f)'(z)}{\alpha z^2(D_\lambda^m f)''(z) + z(D_\lambda^m f)'(z)} - 1\right\}. \tag{3.3}$$

Now, by definition 1.3

$$1 + \frac{zp'(z)}{p(z)} = 1 + \frac{1}{b}\left\{\frac{\alpha z^3(D_\lambda^m f)'''(z) + (1+2\alpha)z^2(D_\lambda^m f)''(z) + z(D_\lambda^m f)'(z)}{\alpha z^2(D_\lambda^m f)''(z) + z(D_\lambda^m f)'(z)} - 1\right\} \prec \phi(z), \quad (0 \leq \alpha < 1). \tag{3.4}$$

Then applying Lemma 2.1 we get the result.
This completes the proof of Theorem 3.1. □

Putting $\lambda = 0$ in Theorem 3.1. Then we have the Ruscheweyh derivative and we get the following new result:

Corollary 3.1. *Let $\phi(z)$ and $F(z)$ be as in Lemma 2.1. The function $f \in H[b, \alpha, m,](\phi)$ if and only if for all $|s| \leq 1$ and $|t| \leq 1$, we have*

$$\left(\frac{s[\alpha z^2(D^m f)''(tz) + z(D^m f)'(tz)]}{t[\alpha z^2(D^m f)''(sz) + z(D^m f)'(sz)]}\right)^{\frac{1}{b}} \prec \frac{sF(tz)}{tF(sz)} \tag{3.5}$$

And putting $m = \lambda = 0$ in Theorem 3.1 gives Theorem 2.1 [8]. Then we have.

Corollary 3.2. *Let $\phi(z)$ and $F(z)$ be as in Lemma 2.1. The function $f \in H[b, \alpha,](\phi)$ if and only if for all $|s| \leq 1$ and $|t| \leq 1$, we have*

$$\left(\frac{s[\alpha z^2 f''(tz) + zf'(tz)]}{t[\alpha z^2 f''(sz) + zf'(sz)]}\right)^{\frac{1}{b}} \prec \frac{sF(tz)}{tF(sz)} \tag{3.6}$$

And putting $\alpha = m = \lambda = 0$ in Theorem 3.1 gives a result in Definition 1.1 [5]. Then we have

Corollary 3.3. *Let $\phi(z)$ and $F(z)$ be as in Lemma 2.1. The function $f \in H[b](\phi)$ if and only if for all $|s| \leq 1$ and $|t| \leq 1$, we have*

$$(\frac{sf'(tz)}{tf'(sz)})^{\frac{1}{b}} \prec \frac{sF(tz)}{tF(sz)} \tag{3.7}$$

Theorem 3.2. *Let ϕ be starlike with respect to 1 and $F(z)$ is given by (2.1) be starlike. If $f \in H[b, \alpha, m, \lambda](\phi)$. Then*

$$\frac{\alpha z^2(D_\lambda^m f)''(z) + z(D_\lambda^m f)'(z)}{z} \prec (\frac{F(z)}{z})^b. \tag{3.8}$$

Proof. Define the functions $p(z)$ and $q(z)$ by

$$p(z) = (\frac{\alpha z^2(D_\lambda^m f)''(z)+z(D_\lambda^m f)'(z)}{z})^{\frac{1}{b}},\ q(z) = \frac{F(z)}{z}$$

Then a computation yields

$$\frac{zp'(z)}{p(z)} = \frac{1}{b}\{\frac{\alpha z^3(D_\lambda^m f)'''(z) + (1+2\alpha)z^2(D_\lambda^m f)''(z) + z(D_\lambda^m f)'(z)}{\alpha z^2(D_\lambda^m f)''(z) + z(D_\lambda^m f)'(z)} - 1\}, \tag{3.9}$$

now, by Definition 1.3 we have

$$1+\frac{zp'(z)}{p(z)} = 1+\frac{1}{b}\{\frac{\alpha z^3(D_\lambda^m f)'''(z) + (1+2\alpha)z^2(D_\lambda^m f)''(z) + z(D_\lambda^m f)'(z)}{\alpha z^2(D_\lambda^m f)''(z) + z(D_\lambda^m f)'(z)} - 1\} \prec \phi(z), \tag{3.10}$$

and

$$\frac{zq'(z)}{q(z)} = \frac{zF'(z)}{F(z)} - 1 = \phi(z) - 1.$$

Since $f \in H[b, \alpha, m, \lambda](\phi)$, we have

$$\frac{zp'(z)}{p(z)} = \frac{1}{b}\{\frac{\alpha z^3(D_\lambda^m f)'''(z) = (1+2\alpha)z^2(D_\lambda^m f)''(z)+z(D_\lambda^m f)'(z)}{\alpha z^2(D_\lambda^m f)''(z)+z(D_\lambda^m f)'(z)} - 1\} \prec \phi(z) - 1 = \frac{zq'(z)}{q(z)}.$$

so

$$\frac{zp'(z)}{p(z)} \prec \frac{zq'(z)}{q(z)}.$$

Now in Lemma 2.2 putting $\varphi(p(z)) = \frac{1}{p(z)}$ and $\varphi(q(z)) = \frac{1}{q(z))}$ we get that
$\frac{zp'(z)}{p(z)} \prec \frac{zq'(z)}{q(z)}$ implies that $p(z) \prec q(z)$
and $(p(z))^b \prec (q(z))^b$
Hence
$\frac{\alpha z^2(D_\lambda^m f)''(z)+z(D_\lambda^m f)'(z)}{z} \prec (\frac{F(z)}{z})^b$.
This completes the proof of Theorem 3.2. □
□

Putting $\lambda = 0$ in Theorem 3.2. Then we have the Ruscheweyh derivative and we get the following new result.

Corollary 3.4. *Let ϕ be starlike with respect to 1 and $F(z)$ is given by (2.1) be starlike. If $f \in H[b, \alpha, m](\phi)$. Then*

$$\frac{\alpha z^2(D^m f)''(z) + z(D^m f)'(z)}{z} \prec (\frac{F(z)}{z})^b. \tag{3.11}$$

And putting $m = \lambda = 0$ in Theorem 3.2 gives Theorem 2.3 [8]. Then we have.

Corollary 3.5. *Let ϕ be starlike with respect to 1 and $F(z)$ is given by (2.1) be starlike. If $f \in H[b,\alpha](\phi)$. Then*

$$\frac{\alpha z^2 f''(z)+zf'(z)}{z} \prec (\frac{F(z)}{z})^b$$

Also putting $\alpha = m = \lambda = 0$ in Theorem 3.2 we have the following new result.

Corollary 3.6. *Let ϕ be starlike with respect to 1 and $F(z)$ is given by (2.1) be starlike. If $f \in H[b](\phi)$. Then*

$$f'(z) \prec (\frac{F(z)}{z})^b$$

3.2. **Coefficients Estimates.** This section is about the class β-convex functions involving complex order defined as follows.

Definition 3.7. Let $b \neq 0$ be a complex number. Let $\phi(z)$ be an analytic function with positive real part on U with $\phi(0) = 1$, $\phi'(0) > 0$ which maps the unit disk U onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Then the class $H[b,\beta,m,\lambda](\phi)$ consists of all analytic functions $f \in A$ satisfying

$$1+\frac{1}{b}\{(1-\beta)(\frac{z(D_\lambda^m f)'(z)}{(D_\lambda^m f)(z)})+\beta(1+\frac{z(D_\lambda^m f)''(z)}{(D_\lambda^m f)'(z)})-1\} \prec \phi(z),\ 0 \leq \beta \leq 1, \lambda \geq o, m > -1.$$

We note that, for $m = \lambda = 0$ we get $H[b,\beta,m,\lambda](\phi) \equiv M_{\beta,b}(\phi)$, [8].

To prove our main result of this section, we need the following:

Lemma 3.8. [5] *If $p(z) = 1 + c_1 z + c_2 z^2 + ...$ is a function with positive real part. Then*

$$|c_2 - \mu c_1^2| \leq 2max\{1, |2\mu - 1|\},$$

and the result is sharp for the functions given by

$$p(z) = \frac{1+z^2}{1-z^2},\quad p(z) = \frac{1+z}{1-z}.$$

Our main result is the following

Theorem 3.3. *Let $0 \leq \beta \leq 1$. Further let $\phi(z) = 1 + B_1 z + B_2 z^2 + ..., z \in U$, where $B_n's$ are real with $B_1 > 0$ and $B_2 \geq 0$. If $f(z)$ given by (1.1) belongs to $H[b,\beta,m,\lambda](\phi)$. Then*

$$|a_3 - \mu a_2^2| \leq$$
$$\frac{B_1|b|}{2(m+1)(m+2)(1+2\beta)(1+2\lambda)} max\{1, |\frac{B_2}{B_1} + \frac{bB_1}{(1+\beta)^2}(1+3\beta - \mu\frac{(m+2)(1+2\lambda)(1+2\beta)}{(m+1)(1+\lambda)^2})c_1^2|\}.$$

Proof. If $f(z) \in H[b,\beta,m,\lambda](\phi)$, then there is a Schwarz function $w(z)$, analytic in U with $w(0) = 0$ and $|w(z)| < 1$ in U such that

$$1+\frac{1}{b}\{(1-\beta)(\frac{z(D_\lambda^m f)'(z)}{(D_\lambda^m f)(z)})+\beta(1+\frac{z(D_\lambda^m f)''(z)}{(D_\lambda^m f)'(z)})-1\} = \phi(w(z)). \tag{3.12}$$

Define $p_1(z)$ by

$$p_1(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + \ldots. \tag{3.13}$$

Since $w(z)$ is a Schwarz function, we see that $Re(p_1(z)) > 0$ and $p_1(0) = 1$. Define the function $p(z)$ by

$$p(z) = 1+\frac{1}{b}\{(1-\beta)(\frac{z(D_\lambda^m f)'(z)}{(D_\lambda^m f)(z)})+\beta(1+\frac{z(D_\lambda^m f)''(z)}{(D_\lambda^m f)'(z)})-1\}$$
$$= 1 + b_1 z + b_2 z^2 + \ldots . \tag{3.14}$$

From (3.13) we get

$$w(z)=\frac{p_1(z)-1}{p_1(z)+1}=\frac{1}{2}[c_1z+(c_2-\frac{c_1^2}{2})z^2+(c_3-\frac{c_1^3}{4}-c_1c_2)z^3+...]\ . \tag{3.15}$$

Since

$$\phi(z)=1+B_1z+B_2z^2+...,z\in U,$$

so, we get

$$\phi(\frac{p_1(z)-1}{p_1(z)+1})=1+\frac{1}{2}B_1c_1z+[\frac{1}{2}B_1(c_2-\frac{c_1^2}{2})+\frac{B_2c_1^2}{4}]z^2+...\ . \tag{3.16}$$

Using (3.12),(3.14) and (3.15) we have

$$p(z)=\phi(\frac{p_1(z)-1}{p_1(z)+1}), \tag{3.17}$$

hence

$$1+b_1z+b_2z^2+...=1+\frac{1}{2}B_1c_1z+[\frac{1}{2}B_1(c_2-\frac{c_1^2}{2})+\frac{B_2c_1^2}{4}]z^2+...\ . \tag{3.18}$$

Equating the coefficients in (3.18) we get

$$b_1=\frac{1}{2}B_1c_1, \tag{3.19}$$

$$b_2=\frac{1}{2}(B_1(c_2-\frac{1}{2}c_1^2))+\frac{1}{4}B_2c_1^2. \tag{3.20}$$

For $f(z)$ in (1.1) we obtain from (3.14) that

$$\begin{aligned}1+\frac{1}{b}\{&(m+1)(1+\lambda)(1+\beta)a_2z+[(m+1)(m+2)(1+2\lambda)(1+2\beta)a_3-\\&(m+1)^2(1+\lambda)^2(1+3\lambda)a_2^2]z^2+...\}=1+b_1z+b_2z^2+....\end{aligned} \tag{3.21}$$

Equating the coefficients in (3.21) we get

$$a_2=\frac{bb_1}{(m+1)(1+\lambda)(1+\beta)}, \tag{3.22}$$

$$a_3=\frac{bb_2+(m+1)^2(1+\lambda)^2(1+3\beta)a_2^2}{(m+1)(m+2)(1+2\lambda)(1+2\beta)}. \tag{3.23}$$

By applying (3.19) and (3.20) in (3.22) and (3.23) respectively we obtain

$$a_2=\frac{bB_1c_1}{2(m+1)(1+\lambda)(1+\beta)}, \tag{3.24}$$

$$\begin{aligned}a3=&\frac{bB_1c_2}{2(m+1)(m+2)(1+2\lambda)(1+2\beta)}+\\&\frac{c_1^2}{4(m+1)(m+2)(1+2\lambda)}[\frac{1+3\beta}{(1+\beta)^2}b^2B_1^2-b(B_1-B_2)].\end{aligned} \tag{3.25}$$

Now, we have

$$a_3-\mu a_2^2=\frac{bB_1}{2(m+1)(m+2)(1+2\lambda)(1+2\beta)}[c_2-\nu c_1^2], \tag{3.26}$$

where

$$\nu=\tfrac{1}{2}[1-\tfrac{B_2}{B_1}-\tfrac{bB_1}{(1+\beta)^2}(1+3\beta-\mu\tfrac{(m+2)(1+2\lambda)(1+2\beta)}{(m+1)(1+\lambda)^2})].$$

Then, applying lemma 3.8 on (3.26) we obtain

$$|a_3 - \mu a_2^2| \leq \frac{B_1|b|}{2(m+1)(m+2)(1+2\beta)(1+2\lambda)}|c_2 - \nu c_1^2| \leq \frac{B_1|b|}{2(m+1)(m+2)(1+2\beta)(1+2\lambda)} 2max\{1, |2\nu - 1|\},$$

for

$$\nu = \frac{1}{2}[1 - \frac{B_2}{B_1} - \frac{bB_1}{(1+\beta)^2}(1 + 3\beta - \mu\frac{(m+2)(1+2\lambda)(1+2\beta)}{(m+1)(1+\lambda)^2})].$$

This completes the proof of Theorem 3.3. □

□

Putting $m = \lambda = 0$ in Theorem 3.3 gives Theorem 3.3 in [8]. Then we have

Corollary 3.9. *Let* $0 \leq \beta \leq 1$. *Further let* $\phi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + ..., z \in U$, *where* $B_n's$ *are real with* $B_1 > 0$ *and* $B_2 \geq 0$. *If* $f(z)$ *given by (1.1) belongs to* $H[b, \beta](\phi)$.*Then*

$$|a_3 - \mu a_2^2| \leq \frac{B_1|b|}{2(1+2\beta)} max\{1, |\frac{B_2}{B_1} + (1 - 2\mu + \beta(3 - 4\mu))\frac{bB_1}{(1+\beta)^2}|\}.$$

Putting $m = \lambda = \beta = 0$ in Theorem 3.3, gives a result obtained in [5]. Then we have

Corollary 3.10. *Let* $0 \leq \beta \leq 1$. *Further let* $\phi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + ..., z \in U$, *where* $B_n's$ *are real with* $B_1 > 0$ *and* $B_2 \geq 0$. *If* $f(z)$ *given by (1.1) belongs to* $H[b](\phi)$. *Then*

$$|a_3 - \mu a_2^2| \leq \frac{B_1|b|}{2} max\{1, |\frac{B_2}{B_1} + (1 - 2\mu)bB_1|\}.$$

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