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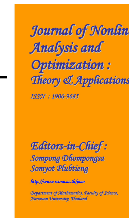

ON THE COMPUTATION OF FIXED POINTS FOR RANDOM OPERATOR EQUATIONS

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ABSTRACT. We approximate fixed points of random operator equation on a complete probability space using Newton's method. Error bounds on the distances involved and some applications are also provided in this study.

KEYWORDS : Newton's method; Complete probability space; Semilocal convergence; Fixed point; Random operator equation; Probabilistic contraction mapping principle.
AMS Subject Classification: 49M15 46S50 47S50

1. INTRODUCTION

Many problems in economics, linear programming and physics lead to random matrix equations [14]. Systems of random equations can also be found in the study of random difference and differential equations [10, 15, 18, 25]. Most methods to approximate solutions are iterative and the practice of numerical analysis for finding such solutions is essentially connected to variants of Newton's method [1–4, 6–9, 11, 13, 16–20, 22–24]. Consider the stochastic initial value problem (see for example [17]):

$$\begin{array}{l} dX(t) = \varphi(t, X(t))\, dB(t) + b(t, X(t))\, dt, \quad 0 \le t \le T \\ X(0) = \zeta, \end{array} \tag{1.1}$$

where, $\{X(t), B(t)\}$ is a family of stochastic processes satisfying some properties (see [17, Definition 2.1]). Eq. (1.1) is also known as Ito–type stochastic differential

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equation. We can solve problem (1.1) using the iterative scheme

$$\begin{aligned} X_0(t) &= \zeta \\ X_{n+1}(t) &= X(0) + \int_0^t \varphi(s, X_n(s))\, dB(s) + \int_0^t b(s, X_n(s))\, ds + \\ & \int_0^t \varphi_x(s, X_n(s))\, (X_{n+1}(s) - X_n(s))\, dB(s) + \\ & \int_0^t b_x(s, X_n(s))\, (X_{n+1}(s) - X_n(s))\, ds. \end{aligned} \tag{1.2}$$

Scheme (1.2) is exactly the Newton method for the stochastic problem (1.1). Note that we can write (1.1) in the following form

$$F(Z)(t) = Z(t) - Z(0) - \int_0^t \varphi(t, X(t))\, dB(t) - \int_0^t b(t, X(t))\, dt. \tag{1.3}$$

In this study we are concerned with the problem of approximating a locally unique solution $x_\star$ of the general random operator equation

$$F(x) = 0. \tag{1.4}$$

We use Newton's method to generate a sequence approximating a locally unique solution of a random operator equation on a complete probability space. A brief survey of some of the general algorithms approximating the solutions of random integral equations is presented in [12].

The paper is organized as follows: Section 2 contains the necessary background results and concepts from probabilistic functional analysis. In Section 3 we provide the semilocal convergence analysis of Newton's method which is faster than the modified Newton's method studied by Bharucha–Reid and Kannan in [13]. We also provide computable upper bounds on the distances involved.

2. PRELIMINARIES

In order for us to make the paper as self contained as possible, we need to introduce some basic concepts and results from probabilistic functional analysis. We refer the reader to [1, 11, 13, 18–20] for more material in this area.

Let $(\Omega, \mathcal{C}, m)$ be a probability measure space and let $(\mathcal{X}, \mathcal{B})$ be a measurable space, where $\mathcal{X}$ is a Banach space and $\mathcal{B}$ is the σ–algebra of all Borel subsets of $\mathcal{X}$. The set Ω is a nonempty abstract set, whose elements ω are called elementary events. $\mathcal{C}$ is a σ–algebra of subsets of Ω. That is, $\mathcal{C}$ is a nonempty class of subsets of Ω satisfying the conditions:

(1) $\Omega \in \mathcal{C}$;
(2) If $A_i \in \mathcal{C}$ $(i = 1, 2)$, then $A_1 - A_2 \in \mathcal{C}$;
(3) If $A_i \in \mathcal{C}$ $(i \geq 1)$, then $\displaystyle\bigcup_{i=1}^{\infty} A_i \in \mathcal{C}$.

The elements of $\mathcal{C}$ are called events. We denote by m a probability measure on $\mathcal{C}$. That is, m is a set function, with domain $\mathcal{C}$, which is nonnegative, countably additive and such that $m(A) \in [0, 1]$ for all $A \in \mathcal{C}$, with $m(\Omega) = 1$. In this study, we assume that m is a complete probability measure. That is m is such that, if $A \in \mathcal{C}$, $m(A) = 0$ and $A_1 \subseteq A$ then $A_1 \in \mathcal{C}$.

A mapping $Q : \Omega \longrightarrow \mathcal{X}$ is said to be a random variable with values in $\mathcal{X}$, if the inverse image under the mapping Q of every Borel set B_0 belongs to $\mathcal{C}$. Let $Q_1(\omega)$

and $Q_2(\omega)$ be $\mathcal{X}$–valued random variables defined on the same probability space. $Q_1(\omega)$ and $Q_2(\omega)$ are said to be equivalent if for every $B_0 \in \mathcal{B}$, we have

$$m(\{\omega \,:\, Q_1(\omega) \in B_0\}\Delta\{\omega \,:\, Q_2(\omega) \in B_0\}) = 0.$$

If $\mathcal{X}$ is separable, then $m(\{\omega \,:\, Q_1(\omega) \neq Q_2(\omega)\}) = 0$.

$Q(\omega)$ is said to be a bounded random operator if there exists a nonnegative real–valued random variable $K(\omega)$ such that for all $x, y \in \mathcal{X}$,

$$\| Q(\omega)x - Q(\omega)y \| \leq K(\omega) \; \| x - y \|, \quad \text{almost surely (a.s.)}$$

A sequence of bounded linear random operators $L_n(\omega)$ is said to be strongly convergent to a bounded linear operator $L_0(\omega)$, if for any $x \in \mathcal{X}$

$$m(\{\omega \,:\, \lim_{n\to\infty} \| L_n(\omega)x - L_0(\omega)x \| = 0\}) = 1.$$

An operator equation

$$Q(\omega)x = y(\omega), \tag{2.1}$$

where, $y(\omega)$ is a given $\mathcal{X}$–valued random variable and $Q(\omega)$ is a given random operator on $\mathcal{X}$ is said to be a random operator equation; and for any $\mathcal{X}$–valued random variable $x_\star(\omega)$ satisfying

$$m(\{\omega \,:\, Q(\omega)x_\star(\omega) = y(\omega)\}) = 1 \tag{2.2}$$

is said to be a random solution of equation (2.1).

If

$$m(\{\omega \,:\, Q(\omega)x_\star(\omega) = x_\star(\omega)\}) = 1 \tag{2.3}$$

then $x_\star(\omega)$ is a random fixed point of random operator equation

$$Q(\omega)x(\omega) = x(\omega). \tag{2.4}$$

We also need the following results on random contraction mappings, fixed points and inverses of random operators. Let $Q(\omega) \,:\, \Omega \times \mathcal{X} \longrightarrow \mathcal{X}$ and let $\ell(\omega)$ be a nonnegative real–valued random variable, such that $m(\{\omega \,:\, \ell(\omega) < 1\}) = 1$. A random operator $Q(\omega)$ on $\mathcal{X}$ is said to be a random contraction operator if

$$m(\{\omega \,: \| Q(\omega)x - Q(\omega)y \| \leq \ell(\omega) \; \| x - y \|\}) = 1 \quad \mathit{for\ all} \quad x, y \in \mathcal{X}. \tag{2.5}$$

We have the following two extentions of the Banach contraction mapping principle [4, 16] for random fixed point theorems due to Hanš [15] (see also [10, 11, 18]).

Theorem 2.1. *Let $Q(\omega) \,:\, \Omega \times \mathcal{X} \longrightarrow \mathcal{X}$ be a continuous random operator, where $\mathcal{X}$ is a separable Banach space.*

Let $\omega \in \Omega$, $x \in \mathcal{X}$ and define sequence $\{Q^n(\omega)\}$ by

$$\begin{aligned} &Q^1(\omega)x = Q(\omega)x \\ &Q^{n+1}(\omega)x = Q(\omega)(Q^n(\omega)x), \quad n \geq 1. \end{aligned} \tag{2.6}$$

If $Q(\omega)$ satisfies the condition

$$m\left(\bigcup_{i=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{x\in\mathcal{X}} \bigcap_{y\in\mathcal{X}} \left\{ \omega \,: \| Q^n(\omega)x - Q^n(\omega)y \| \leq \left(1 - \frac{1}{i}\right) \; \| x - y \| \right\} \right) = 1, \tag{2.7}$$

then, there exists an $\mathcal{X}$-valued random variable $x_\star(\omega)$, which satisfies (2.3). Moreover, sequence $\{x_n(\omega)\}$ given by

$$x_n(\omega) = Q(\omega)x_{n-1}(\omega), \quad (n \geq 1) \tag{2.8}$$

converges to $x_\star(\omega)$ a.e.

Furthemore, if $y_\star(\omega)$ is another $\mathcal{X}$-valued random variable, which satisfies (2.3), then $x_\star(\omega)$ and $y_\star(\omega)$ are equivalent.

We also have the following consequence of Theorem 2.1.

Proposition 2.2. *Let $Q(\omega)\,:\,\Omega\times\mathcal{X}\longrightarrow\mathcal{X}$ be a continuous random contraction operator, where $\mathcal{X}$ is a separable Banach space. Then, there exists an $\mathcal{X}$-valued random variable $x_\star(\omega)$, which is the unique fixed point of $Q(\omega)$. Moreover, sequence $\{x_n(\omega)\}$ generated by (2.7) converges to $x_\star$ a.s.*

Proof. Set

$$E=\{\omega\,:\,\ell(\omega)<1\}, \tag{2.9}$$

$$P=\{\omega\,:\,Q(\omega)x\text{ is continuous in }x\} \tag{2.10}$$

and

$$G(x,y)=\{\omega\,:\parallel Q(\omega)x-Q(\omega)y\parallel\leq\ell(\omega)\ \parallel x-y\parallel\}. \tag{2.11}$$

The intersections in

$$\bigcap_{x\in\mathcal{X}}\bigcap_{y\in\mathcal{X}}\{G(x,y)\cap E\cup P\}$$

can be replaced by intersections over a countable dense set in $\mathcal{X}$, since $\mathcal{X}$ is separable. It follows that condition (2.7) holds with $n=1$. That completes the proof of Proposition 2.2. □

Let $\mathcal{L}(\mathcal{X})$ be the algebra of bounded linear operator on $\mathcal{X}$. Let $Q(\omega)$ be a random operator with values in $\mathcal{L}(\mathcal{X})$. Then, $Q^{-1}(\omega)$ is the random operator $\mathcal{L}(\mathcal{X})$ mapping $Q(\omega)x$ into x a.s.

$Q(\omega)$ is said to be invertible if $Q^{-1}(\omega)$ exists. If $Q(\omega)$ is an invertible random operator with values in $\mathcal{L}(\mathcal{X})$, then $Q^{-1}(\omega)$ is a random operator with values in $\mathcal{L}(\mathcal{X})$.

3. SEMILOCAL CONVERGENCE OF NEWTON'S METHOD

We need the notion of the Fréchet–derivative of a random operator.

Definition 3.1. *Let $Q(\omega)\,:\,\Omega\times\mathcal{X}\longrightarrow\mathcal{X}$ be a continuous random operator, where $\mathcal{X}$ is a separable Banach space. Assume*

$$\lim_{h\to 0}\frac{Q(\omega)(x_0+h)-Q(\omega)x_0}{h} \tag{3.1}$$

exists. That is we assume that for every $\omega\in\Omega$, the operator $Q(\omega)\,:\,\mathcal{X}\longrightarrow\mathcal{X}$ is differentiable. The $\mathcal{X}$-valued element given by (3.1) and denoted by $Q'(\omega)x_0\,:\,\Omega\times\mathcal{X}\longrightarrow\mathcal{X}$ is the Fréchet-derivative at x_0 of $Q(\omega)$. That is we define:

$$Q'(\omega)x_0=\lim_{h\to 0}\frac{Q(\omega)(x_0+h)-Q(\omega)x_0}{h}. \tag{3.2}$$

The randomness of $Q(\omega)$ implies that $Q'(\omega)x_0$ is random linear operator [13].

Note that the definition of $Q'(\omega)$ is not the same with the deterministic notion of the Fréchet–derivative, where, $Q'x_0\,:\,\mathcal{X}\longrightarrow\mathcal{L}(\mathcal{X})$ [2, 9]. Here, $Q'(\omega)\,:\,\mathcal{X}\longrightarrow\mathcal{X}$. As in [13, Example 3.2, p. 233], let us consider random integral operator on $\mathcal{X}=\mathcal{C}[a,b]$, equipped with the sup–norm:

$$Q(\omega)x=\int_a^b H'_v(t,\theta,x(\theta),\omega)\,d\theta, \tag{3.3}$$

where, $H(t,\theta,v,\omega)$ is measurable; $H(t,\theta,v,\omega)$ and $H'_v(t,\theta,u,\omega)$ are jointly continuous for $a\leq t,\ \theta\leq b,\ |v|\leq R,\ R\geq 0$.

Then $Q(\omega) : U(0,R) = \{x :\| x \|\leq R\} \longrightarrow \mathcal{X}$ is Fréchet–differentiable for all $\omega \in \Omega$. It follows from (3.3) that

$$Q'(\omega)x_0[q] = \int_a^b H'_v(t,\theta,x_0(\theta),\omega)\, q(\theta)\, d\theta. \tag{3.4}$$

Let $\mathcal{D}$ be an open subset of $\mathcal{X}$ and $Q(\omega) : \Omega\times\mathcal{D} \longrightarrow \mathcal{X}$ be a random nonlinear operator. Let $Q(\omega)$ be continuous Fréchet–differentiable a.s. Let $x(\omega) : \Omega \longrightarrow \mathcal{D}$ be a $\mathcal{X}$–fixed random variable, such that $(\mathcal{I} - Q'(\omega)\, x(\omega))^{-1} : \Omega\times\mathcal{X} \longrightarrow \mathcal{X}$ is defined and bounded. Clearly, $(\mathcal{I} - Q'(\omega)\, x(\omega))^{-1}$ is a random bounded linear operator, since $Q'(\omega)x$ is random [15, 21].

In order for us to simplify the notation, we denote

$$F = F(\omega) = \mathcal{I} - Q(\omega), \quad x = x(\omega)$$

$$F'(x) = F'(\omega)x(\omega) = \mathcal{I} - Q'(\omega)x(\omega)$$

and

$$F'(x)^{-1} = (F'(\omega)x(\omega))^{-1} = (\mathcal{I} - Q'(\omega)x(\omega))^{-1}.$$

With this notation, we shall use Newton's method (NM)

$$x_{n+1} = x_n - F'(x_n)^{-1}\, F(x_n), \quad (n\geq 0), \quad (x_0 \in \mathcal{D}) \tag{3.5}$$

to generate a sequence $\{x_n\}$ converging to a zero $x_\star = x_\star(\omega)$ of F.

Note also that $x_\star$ is a fixed point of $Q(\omega)$ satisfying (2.4). A semilocal convergence result was given in [13, p. 234] for the modified Newton method (MNM)

$$x_{n+1} = x_n - F'(x_0)^{-1}\, F(x_n), \quad (n\geq 0), \quad (x_0 \in \mathcal{D}). \tag{3.6}$$

Here, we are motivated by optimization considerations and the work of Bharucha–Reid, Kannan [13] on (MNM). We provide a semilocal convergence result for (NM), which a faster than (MNM).

Theorem 3.2. *Let $Q(\omega) : \Omega\times\mathcal{D} \longrightarrow \mathcal{X}$ be a continuous Fréchet-differentiable a.s. random nonlinear operator. Let $x = x(\omega) : \Omega \longrightarrow \mathcal{D}$ be a $\mathcal{X}$-valued random variable, such that $F'(x)^{-1} = (\mathcal{I} - Q'(\omega)x(\omega))^{-1} : \Omega\times\mathcal{X} \longrightarrow \mathcal{X}$ is defined and bounded. Let $x_0 = x_0(\omega) \in \mathcal{D}$ be a fixed $\mathcal{X}$-valued random variable. Then, there exists $U(x_0, r)$ for x_0 and $r > 0$, such that if*

$$\| F'(x)^{-1}\, F'(x_0) \|\leq a(\omega) = a \tag{3.7}$$

for all $x \in \mathcal{D}$ and for some real-valued random variable $a(\omega)$;

$$\| F'(x_0)^{-1}\, F(x_0) \|\leq r\,(1-\ell); \tag{3.8}$$

$$a\, \| F'(x_0)^{-1}\,(F'(x) - F'(x_0)) \|\leq \ell(\omega) \quad for\ all\ x\in\mathcal{D}; \tag{3.9}$$

and

$$\overline{U}(x_0, r) \subseteq \mathcal{D}. \tag{3.10}$$

Then, sequence $\{x_n\}$ $(n\geq 0)$ generated by (NM) (3.5) is well defined, remains in $\overline{U}(x_0,r)$ for all $n\geq 0$ and converges to a unique solution $x_\star = x_\star(\omega)$ of equation $F(x) = 0$ in $\overline{U}(x_0, r)$.

Moreover, the following error estimates hold for all $n\geq 1$:

$$\| x_{n+1} - x_n \|\leq \ell^n\ \| x_n - x_{n-1} \| \tag{3.11}$$

and

$$\| x_n - x_\star \|\leq \frac{\ell^n}{1-\ell}\ \| x_1 - x_0 \|\,. \tag{3.12}$$

Proof. Using (3.5) for $n = 0$ and (3.8), we get x_1 is well defined and $x_1 \in \overline{U}(x_0, r)$. Let us assume $x_k \in \overline{U}(x_0, r)$ for all $k \leq n$. Then x_{k+1} is well defined by (3.5). We shall show $x_{k+1} \in \overline{U}(x_0, r)$. In view of (3.5), we obtain in turn the approximation

$$\begin{aligned}
&\| x_{k+1} - x_k \| = \| F'(x_k)^{-1} F(x_k) \| \\
&= \| F'(x_k)^{-1} (F(x_k) - F(x_{k-1}) + F(x_{k-1})) \| \\
&= \| F'(x_k)^{-1} (F(x_k) - F(x_{k-1}) - F'(x_k)(x_k - x_{k-1})) \| \\
&= \| F'(x_k)^{-1} (F(x_k) - F(x_{k-1}) - F'(x_0)(x_k - x_{k-1})) + \\
&(F'(x_0) - F'(x_k))(x_k - x_{k-1}) \| \\
&= \Big\| (F'(x_k)^{-1} F'(x_0)) \Big(F'(x_0)^{-1} (F(x_k) - F(x_{k-1}) - F'(x_0)(x_k - x_{k-1})) + \\
&F'(x_0)^{-1} (F'(x_0) - F'(x_k))(x_k - x_{k-1}) \Big) \Big\|.
\end{aligned} \tag{3.13}$$

Now, since $Q(\omega)$, $F = \mathcal{I} - Q(\omega)$ are continuously Fréchet–differentiable a.s., we denote by G the set of all $\omega \in \Omega$, such that if $x = x(\omega)$, $y = y(\omega)$ belong in $U(x_0, r)$, then by (3.2)

$$a \ \| F'(x_0)^{-1} (F(y) - F(x) - F'(x_0)(y - x)) \| \leq \epsilon \ \| y - x \|, \tag{3.14}$$

where, $\epsilon > 0$ is such that

$$a (\| F'(x_0)^{-1} (F(x) - F(x_0)) \| + \epsilon) < \ell. \tag{3.15}$$

In particular for $y = x_k$ and $x = x_{k-1}$, we have by (3.14) and (3.15)

$$a \ \| F'(x_0)^{-1} (F(x_k) - F(x_{k-1}) - F'(x_0)(x_k - x_{k-1})) \| \leq \epsilon \ \| x_k - x_{k-1} \|, \tag{3.16}$$

and

$$a (\| F'(x_0)^{-1} (F(x_k) - F(x_0)) \| + \epsilon) < \ell. \tag{3.17}$$

Then , using (3.14), (3.16) and (3.17), we get

$$\| x_{k+1} - x_k \| \leq \ell \ \| x_k - x_{k-1} \|, \tag{3.18}$$

so,

$$\| x_{k+1} - x_k \| \leq \| x_k - x_{k-1} \| \leq \cdots \leq \ell^k \ \| x_1 - x_0 \| \tag{3.19}$$

and by (3.8)

$$\begin{aligned}
\| x_{k+1} - x_0 \| &\leq \sum_{i=1}^{i=k+1} \| x_i - x_{i-1} \| \\
&\leq \sum_{i=0}^{i=k} \ell^k \ \| x_1 - x_0 \| \\
&= \frac{1 - \ell^{k+1}}{1 - \ell} \ \| x_1 - x_0 \| \leq \frac{\| x_1 - x_0 \|}{1 - \ell} \leq r,
\end{aligned} \tag{3.20}$$

which implies $x_{k+1} \in \overline{U}(x_0, r)$. It follows from (3.18) that $\{x_n\}$ is Cauchy in a complete space $\mathcal{X}$ and as such it converges to some $x_\star \in \overline{U}(x_0, r)$.

Moreover, define

$$E = \{\omega \ : \ \ell(\omega) < 1\}, \tag{3.21}$$

and

$$P = \{\omega \ : \ Q(\omega)x \text{ is continuous in } x\}. \tag{3.22}$$

Then, by Proposition 2.2, there exists an $\mathcal{X}$–valued random variable $x_\star$, which is the unique solution of equation $F(x) = 0$ in $\overline{U}(x_0, r)$. Then, sequence $\{x_n\}$ converges to $x_\star$ a.s.

Furthemore, we have for all $i \geq 0$:

$$\| x_{k+i} - x_k \| \leq \sum_{j=k+i-1}^{j=k} \| x_{j+1} - x_j \| \leq \frac{1-\ell^i}{1-\ell} \, \ell^k \, \| x_1 - x_0 \| \, . \tag{3.23}$$

By letting $i \longrightarrow \infty$, we get

$$\| x_\star - x_k \| \leq \frac{\ell^k}{1-\ell} \, \| x_1 - x_0 \| \leq r. \tag{3.24}$$

That completes the proof of Theorem 3.2. □

In the case of the (MNM) (3.6), we can let $a(\omega) = 1$ in Theorem 3.2, so that conditions (3.7) and (3.9) are satisfied. Hence, we arrive at the following Corollary of Theorem 3.2 for the semilocal convergence of (MNM). This result was also essentially (without all the details in the proof) given in [13, p. 234].

Corollary 3.3. *Let $Q(\omega) : \Omega \times \mathcal{D} \longrightarrow \mathcal{X}$ be a continuous Fréchet-differentiable a.s. random nonlinear operator. Let $x_0 = x_0(\omega) : \Omega \longrightarrow \mathcal{D}$ be a $\mathcal{X}$-valued random variable, such that $F'(x_0)^{-1} = (\mathcal{I} - Q'(\omega) x_0(\omega))^{-1} : \Omega \times \mathcal{X} \longrightarrow \mathcal{X}$ is defined and bounded. Let $\ell = \ell(\omega) : \Omega \longrightarrow (0,1)$ be any real valued random variable. Then, there exists $U(x_0, r)$, such that if*

$$\| F'(x_0)^{-1} F(x_0) \| \leq r \, (1-\ell) \quad \textit{and} \quad \overline{U}(x_0, r) \subseteq \mathcal{D}.$$

Then, The conclusions of Theorem 3.2 hold for sequence $\{x_n\}$ $(n \geq 0)$ generated by (MNM) (3.6).

As in the deterministic case [4], [6]–[9], [16, 22], the error estimates (3.11) and (3.12) can be improved so that the usual quadratic convergence of (NM) can be attained. However additional hypothesis of "Lipschitz–type" are needed. We provide such a case here, but first we need the following result on majorizing sequence for (NM).

Lemma 3.4. *[5, 6, 8] Assume that there exist constants $L_0 \geq 0$, $L \geq 0$, with $L_0 \leq L$ and $\eta \geq 0$, such that:*

$$q_{AH} = \overline{L} \, \eta \leq \frac{1}{2}, \tag{3.25}$$

where,

$$\overline{L} = \frac{1}{8} \left(L + 4 \, L_0 + \sqrt{L^2 + 8 \, L_0 \, L} \right). \tag{3.26}$$

The inequality in (3.25) is strict, if $L_0 = 0$.

Then, sequence $\{t_k\}$ $(k \geq 0)$ given by

$$t_0 = 0, \quad t_1 = \eta, \quad t_{k+1} = t_k + \frac{L \, (t_k - t_{k-1})^2}{2 \, (1 - L_0 \, t_k)} \qquad (k \geq 1), \tag{3.27}$$

is well defined, nondecreasing, bounded from above by $t^{\star\star}$, and converges to its unique least upper bound $t^\star \in [0, t^{\star\star}]$, where

$$t^{\star\star} = \frac{2 \, \eta}{2 - \delta}, \tag{3.28}$$

$$1 \leq \delta = \frac{4 \, L}{L + \sqrt{L^2 + 8 \, L_0 \, L}} < 2 \quad \text{for } L_0 \neq 0. \tag{3.29}$$

Moreover, the following estimates hold:

$$L_0 \, t^\star \leq 1, \tag{3.30}$$

$$0 \leq t_{k+1} - t_k \leq \frac{\delta}{2}\,(t_k - t_{k-1}) \leq \cdots \leq \left(\frac{\delta}{2}\right)^k \eta, \quad (k \geq 1), \tag{3.31}$$

$$t_{k+1} - t_k \leq \left(\frac{\delta}{2}\right)^k (2\,q_{AH})^{2^k-1}\,\eta, \quad (k \geq 0), \tag{3.32}$$

$$0 \leq t^\star - t_k \leq \left(\frac{\delta}{2}\right)^k \frac{(2\,q_{AH})^{2^k-1}\,\eta}{1-(2\,q_{AH})^{2^k}}, \quad (2\,q_{AH} < 1), \quad (k \geq 0). \tag{3.33}$$

Then, as in Theorem 3.2, we can show the following semilocal result for the quadratic convergence of (NM).

Theorem 3.5. *Let $Q(\omega) : \Omega \times \mathcal{D} \longrightarrow \mathcal{X}$ be a continuous Fréchet-differentiable a.s. random nonlinear operator. Let $x = x(\omega) : \Omega \longrightarrow \mathcal{D}$ be a $\mathcal{X}$-valued random variable, such that $F'(x)^{-1} = (F'(\omega)x(\omega))^{-1} = (\mathcal{I} - Q'(\omega)x(\omega))^{-1} : \Omega \times \mathcal{X} \longrightarrow \mathcal{X}$ is defined and bounded. Let $x_0 = x_0(\omega) : \Omega \longrightarrow \mathcal{D}$ be a fixed $\mathcal{X}$-valued random variable.*

Assume:

$$\| F'(x_0)^{-1}\,F(x_0) \| \leq \eta(\omega) = \eta; \tag{3.34}$$

for any $x, y \in \mathcal{D}$, $\overline{L_0} = \overline{L_0}(\omega) = L_0(\omega)\ \| x - x_0 \| = L\ \| x - x_0 \| : \Omega \longrightarrow (0,1)$ and $\overline{L} = \overline{L}(\omega) = \dfrac{L(\omega)}{2}\ \| y - x \| = \dfrac{L}{2}\ \| y - x \| : \Omega \longrightarrow (0,1)$ are real-valued random variables;

$$\| F'(x_0)^{-1}\,(F'(x) - F'(x_0)) \| \leq L_0(\omega)\ \| x - x_0 \|; \tag{3.35}$$

$$\| F'(x)^{-1}\,F'(x_0) \| \leq \frac{1}{1 - L_0(\omega)\ \| x - x_0 \|}; \tag{3.36}$$

$$\| F'(x_0)^{-1}\,(F(y) - F(x) - F'(x)\,(y - x)) \| \leq \frac{L(\omega)}{2}\ \| y - x \|^2; \tag{3.37}$$

hypotheses of Lemma 3.4 hold with $\eta(\omega) = \eta$, $L_0(\omega) = L_0$ and $L(\omega) = L$;

and

$$\overline{U}(x_0, t^\star) \subseteq \mathcal{D}, \tag{3.38}$$

where $t^\star$ is given in Lemma 3.4.

Then, sequence $\{x_n\}$ $(n \geq 0)$ generated by (NM) (3.5) is well defined, remains in $\overline{U}(x_0, t^\star)$ for all $n \geq 0$ and converges to a unique solution $x_\star = x_\star(\omega)$ of equation $F(x) = 0$ in $\overline{U}(x_0, t^\star)$.

Moreover, the following error estimates hold for all $n \geq 1$:

$$\| x_{n+1} - x_n \| \leq t_{n+1} - t_n \tag{3.39}$$

and

$$\| x_n - x_\star \| \leq t^\star - t_n. \tag{3.40}$$

Remark 3.6. *Note that in view of Lemma 3.4, the convergence order of sequence $\{x_n\}$ is quadratic for $q_{AH} < 1/2$ and linear if $q_{AH} = 1/2$.*

Proof. (of Theorem 3.5) We follow the proof of Theorem 3.2. But this time using (3.5), (3.27), (3.34)–(3.37) and the approximation

$$F'(x_0)^{-1}\,F(x_k) = F'(x_0)^{-1}\,(F(x_k) - F(x_{k-1}) - F'(x_{k-1})\,(x_k - x_{k-1})),$$

we get that

$$\begin{aligned} &\| x_{k+1} - x_k \| \\ &\leq \| F'(x_k)^{-1}\,F'(x_0) \| \| F'(x_0)^{-1}\,(F(x_k) - F(x_{k-1}) - F'(x_{k-1})\,(x_k - x_{k-1})) \| \\ &\leq \frac{1}{1 - L_0\ \| x_k - x_0 \|}\,\frac{L}{2}\ \| x_k - x_{k-1} \|^2 \leq \frac{L\,(t_k - t_{k-1})^2}{1 - L_0\,(1 - L_0\,t_k)^2} = t_{k+1} - t_k. \end{aligned} \tag{3.41}$$

Hence, $\{t_k\}$ $(k \geq 0)$ is a majorizing sequence for $\{x_k\}$. The rest follows as in Theorem 3.2 and the deterministic case [4], [6]–[9]. That completes the proof of Theorem 3.5. □

As an example, the results obtained here find applications in the solution of equations of the form

$$Q\,x = z(t,\omega),$$
$$(Q - \lambda\,\mathcal{I})\,x = z(t,\omega),$$
$$Q(\omega)\,x = z(t),$$
$$(Q(\omega) - \lambda\,\mathcal{I})\,x = z(t),$$
$$Q(\omega)\,x = z(t,\omega)$$

and

$$(Q(\omega) - \lambda\,\mathcal{I})\,x = z(t,\omega),$$

where, the imput is a random function. Then, "Lipschitz–type" estimates (3.35)–(3.37) can be realized using inversion theorems, which can be found , e.g. in [9], [13].

Applications and examples, including the solution of nonlinear Chandrasekhar–type integral equations appearing in radiative transfer are also find in [4], [6]–[9].

CONCLUSION

We provided new convergence results for (NM) and (MNM) for solving random operator equations. The sufficient convergence conditions are obtained using Lipschitz and center–Lipschitz conditions instead of the only Lipschitz condition used in [13]. Our results extend the applicability of this method studied in [13]. Some remarks and applications in the deterministic case are also provided in this study.

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