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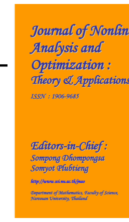

SQUARE ROOT AND 3RD ROOT FUNCTIONAL EQUATIONS IN C^*-ALGEBRAS: AN FIXED POINT APPROACH

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ABSTRACT. In this paper, we introduce a square root functional equation and a 3rd root functional equation. Using fixed point methods, we prove the Hyers-Ulam stability of the square root functional equation and of the 3rd root functional equation in C^*-algebras.

KEYWORDS : Hyers-Ulam stability; C^*-algebra; Convex cone; Fixed point, Square root functional equation; 3rd root functional equation.

1. INTRODUCTION AND PRELIMINARIES

The stability problem of functional equations was originated from a question of Ulam [27] concerning the stability of group homomorphisms. Hyers [12] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [26] for linear mappings by considering an unbounded Cauchy difference. The paper of Th.M. Rassias [26] has provided a lot of influence in the development of what we call *generalized Hyers-Ulam stability* or *Hyers-Ulam-Rassias stability* of functional equations. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [10] by replacing the unbounded Cauchy difference by a general control function in the spirit of the Th.M. Rassias' approach. J.M. Rassias [23]–[25] followed the innovative approach of the Th.M. Rassias' theorem [26] in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \cdot \|y\|^q$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$. The stability problems of several functional equations have been extensively investigated by a

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number of authors and there are many interesting results concerning this problem (see [5, 8, 9, 13, 16, 17]).

Definition 1.1. [7] Let A be a C^*-algebra and $x \in A$ a self-adjoint element, i.e., $x^* = x$. Then x is said to be *positive* if it is of the form yy^* for some $y \in A$.

The set of positive elements of A is denoted by A^+.

Note that A^+ is a closed convex cone (see [7]).

It is well-known that for a positive element x and a positive integer n there exists a unique positive element $y \in A^+$ such that $x = y^n$. We denote y by $x^{\frac{1}{n}}$ (see [11]).

In this paper, we introduce a *square root functional equation*

$$S\left(x + y + x^{\frac{1}{4}}y^{\frac{1}{2}}x^{\frac{1}{4}} + y^{\frac{1}{4}}x^{\frac{1}{2}}y^{\frac{1}{4}}\right) = S(x) + S(y) \tag{1.1}$$

and a *3rd root functional equation*

$$T\left(x + y + 3x^{\frac{1}{3}}y^{\frac{1}{3}}x^{\frac{1}{3}} + 3y^{\frac{1}{3}}x^{\frac{1}{3}}y^{\frac{1}{3}}\right) = T(x) + T(y) \tag{1.2}$$

for all $x, y \in A^+$. Each solution of the square root functional equation is called a *square root mapping* and each solution of the 3rd root functional equation is called a *3rd root mapping*.

Note that the functions $S(x) = \sqrt{x} = x^{\frac{1}{2}}$ and $T(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$ in the set of non-negative real numbers are solutions of the functional equations (1.1) and (1.2), respectively.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We recall a fundamental result in fixed point theory.

Theorem 1.1. [2, 6] *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \geq n_0$;

(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*

(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;*

(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ *for all $y \in Y$.*

In 1996, G. Isac and Th.M. Rassias [14] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [3, 4], [19]–[22]).

Using the fixed point method, we prove the Hyers-Ulam stability of the functional equations (1.1) and (1.2) in C^*-algebras.

Throughout this paper, let A^+ and B^+ be the sets of positive elements in C^*-algebras A and B, respectively.

2. STABILITY OF THE SQUARE ROOT FUNCTIONAL EQUATION

In this section, we investigate the square root functional equation in C^*-algebras.

Lemma 2.1. [15] *Let $S : A^+ \to B^+$ be a square root mapping satisfying* (1.1). *Then S satisfies*

$$S(4^n x) = 2^n S(x)$$

for all $x \in A^+$ and all $n \in \mathbb{Z}$.

We prove the Hyers-Ulam stability of the square root functional equation in C^*-algebras. Note that the fundamental ideas in the proofs of the main results in Sections 2 and 3 are contained in [2, 3, 4].

Theorem 2.1. *Let $\varphi : A^+ \times A^+ \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq \frac{L}{2}\varphi(4x, 4y) \tag{2.1}$$

for all $x, y \in A^+$. Let $f : A^+ \to B^+$ be a mapping satisfying

$$\left\| f\left(x + y + x^{\frac{1}{4}} y^{\frac{1}{2}} x^{\frac{1}{4}} + y^{\frac{1}{4}} x^{\frac{1}{2}} y^{\frac{1}{4}}\right) - f(x) - f(y) \right\| \leq \varphi(x, y) \tag{2.2}$$

for all $x, y \in A^+$. Then there exists a unique square root mapping $S : A^+ \to A^+$ satisfying (1.1) *and*

$$\|f(x) - S(x)\| \leq \frac{L}{2 - 2L}\varphi(x, x) \tag{2.3}$$

for all $x \in A^+$.

Proof. Letting $y = x$ in (2.2), we get

$$\|f(4x) - 2f(x)\| \leq \varphi(x, x) \tag{2.4}$$

for all $x \in A^+$.

Consider the set

$$X := \{g : A^+ \to B^+\}$$

and introduce the generalized metric on X:

$$d(g, h) = \inf\{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\| \leq \mu\varphi(x, x), \ \forall x \in A^+\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (X, d) is complete (see [18]).

Now we consider the linear mapping $J : X \to X$ such that

$$Jg(x) := 2g\left(\frac{x}{4}\right)$$

for all $x \in A^+$.

Let $g, h \in X$ be given such that $d(g, h) = \varepsilon$. Then

$$\|g(x) - h(x)\| \leq \varphi(x, x)$$

for all $x \in A^+$. Hence

$$\|Jg(x) - Jh(x)\| = \|2g\left(\frac{x}{4}\right) - 2h\left(\frac{x}{4}\right)\| \leq L\varphi(x, x)$$

for all $x \in A^+$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in X$.

It follows from (2.4) that

$$\|f(x) - 2f\left(\frac{x}{4}\right)\| \leq \frac{L}{2}\varphi(x,x)$$

for all $x \in A^+$. So $d(f, Jf) \leq \frac{L}{2}$.

By Theorem 1.2, there exists a mapping $S : A^+ \to B^+$ satisfying the following:

(1) S is a fixed point of J, i.e.,

$$S\left(\frac{x}{4}\right) = \frac{1}{2}S(x) \tag{2.5}$$

for all $x \in A^+$. The mapping S is a unique fixed point of J in the set

$$M = \{g \in X : d(f,g) < \infty\}.$$

This implies that S is a unique mapping satisfying (2.5) such that there exists a $\mu \in (0,\infty)$ satisfying

$$\|f(x) - S(x)\| \leq \mu\varphi(x,x)$$

for all $x \in A^+$;

(2) $d(J^n f, S) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\longrightarrow\infty} 2^n f\left(\frac{x}{4^n}\right) = S(x)$$

for all $x \in A^+$;

(3) $d(f,S) \leq \frac{1}{1-L}d(f,Jf)$, which implies the inequality

$$d(f,S) \leq \frac{L}{2-2L}.$$

This implies that the inequality (2.3) holds.

By (2.1) and (2.2),

$$\begin{aligned} & 2^n \left\| f\left(\frac{x+y+x^{\frac{1}{4}}y^{\frac{1}{2}}x^{\frac{1}{4}} + y^{\frac{1}{4}}x^{\frac{1}{2}}y^{\frac{1}{4}}}{4^n}\right) - f\left(\frac{x}{4^n}\right) - f\left(\frac{y}{4^n}\right)\right\| \\ \leq \quad & 2^n\varphi\left(\frac{x}{4^n}, \frac{y}{4^n}\right) \leq L^n\varphi(x,y) \end{aligned}$$

for all $x, y \in A^+$ and all $n \in \mathbb{N}$. So

$$\left\|S\left(x+y+x^{\frac{1}{4}}y^{\frac{1}{2}}x^{\frac{1}{4}} + y^{\frac{1}{4}}x^{\frac{1}{2}}y^{\frac{1}{4}}\right) - S(x) - S(y)\right\| = 0$$

for all $x, y \in A^+$. Thus the mapping $S : A^+ \to B^+$ is a square root mapping, as desired. □

Corollary 2.2. *Let $p > \frac{1}{2}$ and θ_1, θ_2 be non-negative real numbers, and let $f : A^+ \to B^+$ be a mapping such that*

$$\begin{aligned} & \left\|f\left(x+y+x^{\frac{1}{4}}y^{\frac{1}{2}}x^{\frac{1}{4}} + y^{\frac{1}{4}}x^{\frac{1}{2}}y^{\frac{1}{4}}\right) - f(x) - f(y)\right\| \\ \leq \quad & \theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}} \end{aligned} \tag{2.6}$$

for all $x, y \in A^+$. Then there exists a unique square root mapping $S : A^+ \to B^+$ satisfying (1.1) *and*

$$\|f(x) - S(x)\| \leq \frac{2\theta_1 + \theta_2}{4^p - 2}||x||^p$$

for all $x \in A^+$.

Proof. The proof follows from Theorem 2.1 by taking $\varphi(x,y) = \theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}}$ for all $x, y \in A^+$. Then we can choose $L = 2^{1-2p}$ and we get the desired result. □

Theorem 2.2. *Let $\varphi : A^+ \times A^+ \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \le 2L\varphi\left(\frac{x}{4}, \frac{y}{4}\right)$$

for all $x, y \in A^+$. Let $f : A^+ \to B^+$ be a mapping satisfying (2.2). *Then there exists a unique square root mapping $S : A^+ \to A^+$ satisfying* (1.1) *and*

$$\|f(x) - S(x)\| \le \frac{1}{2-2L}\varphi(x,x)$$

for all $x \in A^+$.

Proof. Let (X, d) be the generalized metric space defined in the proof of Theorem 2.1.

Consider the linear mapping $J : X \to X$ such that

$$Jg(x) := \frac{1}{2}g(4x)$$

for all $x \in A^+$.

It follows from (2.4) that

$$\left\|f(x) - \frac{1}{2}f(4x)\right\| \le \frac{1}{2}\varphi(x,x)$$

for all $x \in A^+$. So $d(f, Jf) \le \frac{1}{2}$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.3. *Let $0 < p < \frac{1}{2}$ and θ_1, θ_2 be non-negative real numbers, and let $f : A^+ \to B^+$ be a mapping satisfying* (2.6). *Then there exists a unique square root mapping $S : A^+ \to B^+$ satisfying* (1.1) *and*

$$\|f(x) - S(x)\| \le \frac{2\theta_1 + \theta_2}{2 - 4^p}||x||^p$$

for all $x \in A^+$.

Proof. The proof follows from Theorem 2.2 by taking $\varphi(x,y) = \theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}}$ for all $x, y \in A^+$. Then we can choose $L = 2^{2p-1}$ and we get the desired result. □

3. STABILITY OF THE 3RD ROOT FUNCTIONAL EQUATION

In this section, we investigate the 3rd root functional equation in C^*-algebras.

Lemma 3.1. [15] *Let $T : A^+ \to B^+$ be a 3rd root mapping satisfying* (1.2). *Then T satisfies*

$$T(8^n x) = 2^n T(x)$$

for all $x \in A^+$ and all $n \in \mathbb{Z}$.

We prove the Hyers-Ulam stability of the 3rd root functional equation in C^*-algebras.

Theorem 3.1. *Let $\varphi : A^+ \times A^+ \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \le \frac{L}{2}\varphi(8x, 8y)$$

for all $x, y \in A^+$. Let $f : A^+ \to B^+$ be a mapping satisfying

$$\left\| f\left(x + y + 3x^{\frac{1}{3}}y^{\frac{1}{3}}x^{\frac{1}{3}} + 3y^{\frac{1}{3}}x^{\frac{1}{3}}y^{\frac{1}{3}}\right) - f(x) - f(y)\right\| \quad \le \quad \varphi(x, y) \tag{3.1}$$

for all $x, y \in A^+$. Then there exists a unique 3rd root mapping $T : A^+ \to A^+$ satisfying (1.2) *and*

$$\|f(x) - T(x)\| \le \frac{L}{2 - 2L}\varphi(x, x)$$

for all $x \in A^+$.

Proof. Letting $y = x$ in (3.1), we get

$$\|f(8x) - 2f(x)\| \le \varphi(x, x) \tag{3.2}$$

for all $x \in A^+$.

Let (X, d) be the generalized metric space defined in the proof of Theorem 2.1.

Consider the linear mapping $J : X \to X$ such that

$$Jg(x) := 2g\left(\frac{x}{8}\right)$$

for all $x \in A^+$.

Now we consider the linear mapping $J : X \to X$ such that

$$Jg(x) := 2g\left(\frac{x}{8}\right)$$

for all $x \in A^+$.

It follows from (3.2) that

$$\|f(x) - 2f\left(\frac{x}{8}\right)\| \le \frac{L}{2}\varphi(x, x)$$

for all $x \in X$. So $d(f, Jf) \le \frac{L}{2}$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 3.2. *Let $p > \frac{1}{3}$ and θ_1, θ_2 be non-negative real numbers, and let $f : A^+ \to B^+$ be a mapping such that*

$$\begin{aligned} &\left\| f\left(x + y + 3x^{\frac{1}{3}}y^{\frac{1}{3}}x^{\frac{1}{3}} + 3y^{\frac{1}{3}}x^{\frac{1}{3}}y^{\frac{1}{3}}\right) - f(x) - f(y)\right\| \\ \le \quad &\theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}} \end{aligned} \tag{3.3}$$

for all $x, y \in A^+$. Then there exists a unique 3rd root mapping $T : A^+ \to B^+$ satisfying (1.2) *and*

$$\|f(x) - T(x)\| \le \frac{2\theta_1 + \theta_2}{8^p - 2}||x||^p$$

for all $x \in A^+$.

Proof. The proof follows from Theorem 3.1 by taking $\varphi(x, y) = \theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}}$ for all $x, y \in A^+$. Then we can choose $L = 2^{1-3p}$ and we get the desired result. □

Theorem 3.2. *Let* $\varphi : A^+ \times A^+ \to [0, \infty)$ *be a function such that there exists an* $L < 1$ *with*

$$\varphi(x, y) \le 2L\varphi\left(\frac{x}{8}, \frac{y}{8}\right)$$

for all $x, y \in A^+$. *Let* $f : A^+ \to B^+$ *be a mapping satisfying* (3.1). *Then there exists a unique 3rd root mapping* $T : A^+ \to A^+$ *satisfying* (1.2) *and*

$$\|f(x) - T(x)\| \le \frac{1}{2 - 2L}\varphi(x, x)$$

for all $x \in A^+$.

Proof. Let (X, d) be the generalized metric space defined in the proof of Theorem 2.1.

Consider the linear mapping $J : X \to X$ such that

$$Jg(x) := \frac{1}{2}g\,(8x)$$

for all $x \in A^+$.

It follows from (3.2) that

$$\left\| f(x) - \frac{1}{2}f(8x) \right\| \le \frac{1}{2}\varphi(x, x)$$

for all $x \in A^+$. So $d(f, Jf) \le \frac{1}{2}$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 3.3. *Let* $0 < p < \frac{1}{3}$ *and* θ_1, θ_2 *be non-negative real numbers, and let* $f : A^+ \to B^+$ *be a mapping satisfying* (3.3). *Then there exists a unique 3rd root mapping* $T : A^+ \to B^+$ *satisfying* (1.2) *and*

$$\|f(x) - T(x)\| \le \frac{2\theta_1 + \theta_2}{2 - 8^p}||x||^p$$

for all $x \in A^+$.

Proof. The proof follows from Theorem 3.2 by taking $\varphi(x, y) = \theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}}$ for all $x, y \in A^+$. Then we can choose $L = 2^{3p-1}$ and we get the desired result. □

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