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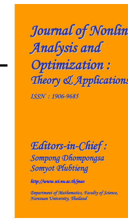

NEW POSSIBILITIES REGARDING THE ALTERNATING PROJECTIONS METHOD

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ABSTRACT. We provide a short survey of some recent results of ours regarding the convergence of infinite products of nonexpansive operators, some of which are orthogonal projections, while others may even be nonlinear. It turns out that the standard requirement of cyclic order of the projections may be replaced with the positivity of the angles between the given subspaces. Moreover, one of these angles may even be unknown, provided the corresponding projection appears in the infinite product sufficiently rarely.

KEYWORDS : Alternating projection; Angle; Infinite product; Nonexpansive operator.

1. INTRODUCTION

The well-known method of alternating projections is used for finding points in the intersection of a finite number of closed and convex sets via individual approximations from each one of these sets. If all the sets are closed linear subspaces of a given Hilbert space H, then we can use the best approximations which in this case are the standard orthogonal projections. Taking an initial point $x_0 \in H$ and projecting it successively onto the given subspaces, we may expect that the sequence of iterations $\{x_n = P_n P_{n-1} \cdots P_1 x_0\}$ will converge to a point x^* in the intersection of all the subspaces.

We are interested in either strong or uniform convergence. The only necessary condition for this is that every subspace is involved in the iteration process infinitely often. The various sufficient conditions presented in the literature require

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a special order of the projections and/or additional information regarding interrelations between the given subspaces. For example, many iteration procedures were studied for the case where the orthogonal projections are arranged cyclically in a prescribed repeated order. In this connection, one should mention, first and foremost, the celebrated pioneering works by J. von Neumann [4] and I. Halperin [3]. For instance, in the case of three projections P_U, P_V, P_W onto three given subspaces U, V, W, respectively, Halperin's theorem ensures the strong convergence of the sequence $\{x_n = (P_U P_V P_W)^n x_0\}$ to $P_{U\cap V\cap W} x_0$ for any $x_0 \in H$.

The uniform convergence of iterations cannot be obtained by only using the order of the projections, because any projection operator has norm 1, which is insufficient for achieving uniform convergence. Nevertheless, the product of two projections $P_U P_V$ could have norm less than 1 in the case of a positive angle between the subspaces U and V. The concept of an "angle between subspaces" in an arbitrary Hilbert space has many different definitions applied to different problems and situations. We adopt the one given by K. Friedrichs in [2] and widely used for studying projection methods in the monograph [1] by F. Deutsch. Namely,

$$\theta(U,V) = \inf\{|\theta(x,y)| : \ x \in U^\circ, \ y \in V^\circ, \ x, y \neq 0\},$$

where $U^\circ = U \cap (U\cap V)^\perp, \ V^\circ = V \cap (U\cap V)^\perp$ and

$$\theta(x,y) = \arccos \frac{\langle x, y\rangle}{\|x\| \cdot \|y\|}$$

is the usual angle between two vectors in a space with inner product $\langle \cdot\,, \cdot\rangle$. The uniform norm convergence of the sequence $\{x_n = (P_U P_V)^n x_0\}$ to $P_{U\cap V} x_0$ immediately follows from the estimate

$$\|P_U P_V x - P_{U\cap V} x\| \le \cos\theta(U,V)\, \|x - P_{U\cap V} x\| \qquad \text{for any } x \in H.$$

The angle $\theta(U,V)$ is positive (and, consequently, $\cos\theta(U,V) < 1$) whenever at least one of the subspaces U, V is finite dimensional. Otherwise the angle may be zero even in rather standard situations.

Example. Let $\{e_n\}_{n=1}^\infty$ be an orthonormal basis in the separable Hilbert space H and let the numerical sequences $\{\alpha_n\}$ and $\{\beta_n\}$ be such that $\alpha_n^2 + \beta_n^2 = 1$ for each n with $\alpha_n \longrightarrow 0$ as $n \longrightarrow \infty$. We consider two infinite dimensional subspaces U and V with the bases $\{u_n\}_{n=1}^\infty$ and $\{v_n\}_{n=1}^\infty$, respectively, defined by

$$u_{2n-1} = \frac{1}{\sqrt{2}}(e_{3n} - e_{3n-1}), \quad u_{2n} = \frac{1}{\sqrt{2}}(e_{3n} + e_{3n-1}), v_n = \alpha_n e_{3n-2} + \beta_n e_{3n}\,, \quad n = 1, 2, \ldots.$$

It is a simple task to check that both bases $\{u_n\}$ and $\{v_n\}$ are orthonormal and that $U \cap V = \{0\}$, so that $U = U^\circ$ and $V = V^\circ$. Hence we may use the vectors $e_{3n} = \frac{1}{\sqrt{2}}(u_{2n-1} + u_{2n})$ from the subspace U and v_n from the subspace V, obtaining that $\cos\theta(U,V) \ge \langle e_{3n}, v_n\rangle = \beta_n \longrightarrow 1$ as $n \longrightarrow \infty$, that is, $\theta(U,V) = 0$.

Observe, in addition, that for any of the basis vectors $u_m \in U^\circ$ and $v_n \in V^\circ$, the angles $\theta(u_m, v_n) > \frac{\pi}{4}$ because $\langle u_m, v_n\rangle < \frac{1}{\sqrt{2}}$ for all combinations of m, n. This means that, generally speaking, the conclusion that $\theta(U,V) = 0$ cannot be obtained by only considering the basis vectors, making the problem of deciding whether $\theta(U,V) > 0$ rather difficult in general.

2. UNIFORM CONVERGENCE

In the case of three subspaces U, V, W the pairwise angles between them are insufficient for getting uniform convergence. The known proofs for the projections arranged as $P_U P_V P_W$ require either the angles $\alpha = \theta(U, V)$ and $\beta = \theta(U \cap V, W)$ or the angles $\alpha = \theta(V, W)$ and $\beta = \theta(U, V \cap W)$. The corresponding estimate is

$$\|P_U P_V P_W x - P_{U\cap V\cap W} x\| \leq q\|x - P_{U\cap V\cap W} x\| \qquad \text{for any } \ x \in H,$$

with $q = (1 - \sin^2 \alpha \sin^2 \beta)^{1/2}$ in the first case and with

$$q = \max \left[\cos \frac{\beta}{2},\ (1 + \sin^2 \frac{\beta}{2} \tan^2 \alpha)^{-1/2}\right]$$

in the second one. We see that uniform convergence may be ascertained only when both α and β are positive. In particular, uniform convergence occurs if at least one of the subspaces is finite dimensional.

Using angles between subspaces, we are able to consider short fragments of the iteration process independently of other fragments, which may be further combined in arbitrary order, making the cyclic order of projections unnecessary:

$$\cdots (P_U P_V P_W)(P_V P_W P_U)(P_W P_V P_U)(P_V P_W P_U) \cdots$$

Moreover, we may insert between these fragments other nonexpansive, even nonlinear, operators if this is needed for improving the calculation process or for obtaining solutions with additional properties. Recall that an operator A is called *nonexpansive* if $\|Ax - Ay\| \leq \|x - y\|$ for any $x, y \in H$.

Generally speaking, the approximation process can be presented as an infinite product of operators $\prod_{n=1}^{\infty} A_n \mathcal{P}_n$, where all A_n are nonexpansive and all $\mathcal{P}_n$ are linear contractions, obtained as compositions of the projections P_U, P_V, P_W in any suitable order.

Theorem 2.1. *Let the subspace $F = U \cap V \cap W$ be invariant under all the nonexpansive operators A_n acting on a Hilbert space H and participating in a given infinite product. Let $q < 1$ be such that*

$$\|\mathcal{P}_n x - P_F x\| \leq q\|x - P_F x\| \qquad \textit{for any } x \in H,\ n = 1, 2, \ldots.$$

Then, for any initial point $x_0 \in H$, the corresponding partial products form a sequence

$$\{x_n = A_n \mathcal{P}_n A_{n-1} \mathcal{P}_{n-1} \cdots A_1 \mathcal{P}_1 x_0\}$$

such that

$$\lim_{n \longrightarrow \infty} \|x_n - P_F x_n\| = 0,$$

uniformly over any bounded set of initial points x_0. If, in addition, all $x \in F$ are fixed points for each A_n, then $\{x_n\}$ is strongly convergent to some $x^* \in F$.

In particular, the operators A_n could be the separate projections P_U, P_V, P_W or their pairwise products, and thus the triple products $\mathcal{P}_n = P_U P_V P_W$ (or in another order) need not follow cyclically and may be located in the iterative process arbitrarily far from each other. In practice this means that the order of the projections may be *essentially random* (not cyclic and even not “almost" cyclic); the only condition for uniform convergence to the best approximation from $U \cap V \cap W$ is that the triple products with known positivity of the corresponding angles appear infinitely many times, for instance,

$$\cdots P_V P_U P_V (P_W P_U P_V) P_W P_V P_W P_V (P_W P_V P_U) P_V P_U \cdots .$$

3. STRONG CONVERGENCE

Rather surprisingly, a similar assertion can be proved in the case where only one angle (say $\theta(U,V)$) is known to be positive and no properties of the angles involving the third subspace W are given. Of course, the convergence of iterations in this case may be only strong, but it holds without imposing any cyclic arrangement of the projections.

Theorem 3.1. *Let U, V, W be three subspaces of a Hilbert space H such that the angle $\theta(U,V)$ is positive. Let the nonexpansive operators A_n, $n = 1, 2, \ldots,$ be such that all elements of the subspace $U \cap V$ are fixed points of each A_n. Let a sequence of natural numbers $\{k_n\}$ be such that*

$$\sum_{n=1}^{\infty} q^{k_n} < \infty, \qquad \text{where} \quad q = \cos\theta(U,V).$$

Then, for any $x \in H$, there exists $x^* \in U \cap V \cap W$ such that

$$\lim_{n \longrightarrow \infty} \|P_W A_n (P_U P_V)^{k_n} P_W A_{n-1} (P_U P_V)^{k_{n-1}} \cdots P_W A_1 (P_U P_V)^{k_1} x - x^*\| = 0.$$

Note that Theorem 3.1 admits an interesting new application to Numerical Analysis. Suppose we are interested in finding the point $P_{U\cap V\cap W} x_0$ for some given $x_0 \in H$. By Halperin's theorem we may use the iterations $x_n = (P_W P_U P_V)^n x_0$ which converge to the sought-after point. Suppose the subspace W is such that any computation of the projection P_W is difficult in comparison with other projections. Omitting all A_n, we see that, in the case where $\theta(U,V) > 0$, Theorem 3.1 enables another iteration process, namely,

$$x_N = P_W (P_U P_V)^{k_n} P_W (P_U P_V)^{k_{n-1}} \cdots P_W (P_U P_V)^{k_1} x_0, \quad N = n + \sum_{i=1}^{n} k_i,$$

which converges to the same point $P_{U\cap V\cap W} x_0$ for arbitrarily quickly increasing k_n and, correspondingly, arbitrarily rare computations of P_W.

We now give a brief sketch of the proof of Theorem 3.1. We will use the following result from [5, p. 1512]. Here $\rho(y, F) := \inf\{\|y - z\| : z \in F\}$.

Proposition 3.1. *Let $T : H \longrightarrow H$ be a nonexpansive operator and let a set $F \subset H$ be such that, for any given $x \in H$, the sequence $\{\rho(T^n x, F)\}$ converges to 0 as $n \longrightarrow \infty$. Let a sequence $\{x_n\}_{n=0}^{\infty} \subset H$ be such that, for each $n = 0, 1, 2, \ldots$,*

$$\|x_{n+1} - T x_n\| \le \gamma_n, \qquad \sum_{n=1}^{\infty} \gamma_n < \infty.$$

Then $\rho(x_n, F) \longrightarrow 0$ as $n \longrightarrow \infty$. If, in addition, $\{T^n x\}$ is strongly convergent for each $x \in H$, then $\{x_n\}$ strongly converges to some $x^* \in F$.

According to the von Neumann theorem for two projections,

$$\lim_{n \longrightarrow \infty} \|(P_W P_{U\cap V})^n x - P_{U\cap V\cap W} x\| = 0, \qquad \forall x \in H,$$

which corresponds to the hypotheses of Proposition 3.1 if we put $T = P_W P_{U\cap V}$ and $F = U \cap V \cap W$. Next, for arbitrary $x \in H$, we define $x_1 = x$, $x_{n+1} = P_W A_n (P_U P_V)^{k_n} x_n$. The inequality

$$\|P_U P_V x - P_{U\cap V} x\| \le q \,\|x - P_{U\cap V} x\|, \quad q = \cos\theta(U, V),$$

which has already been indicated above, can be readily generalized by induction to

$$\|(P_U P_V)^n x - P_{U\cap V} x\| \le q^n \,\|x - P_{U\cap V} x\|,$$

and by the postulated properties of the operators A_n, we have $A_n P_{U\cap V} x = P_{U\cap V} x$ for each n. Consequently,

$$\begin{gathered}\|x_{n+1} - T x_n\| = \|P_W A_n (P_U P_V)^{k_n} x_n - P_W P_{U\cap V} x_n\| \\ \le \|A_n (P_U P_V)^{k_n} x_n - A_n P_{U\cap V} x_n\| \le \|(P_U P_V)^{k_n} x_n - P_{U\cap V} x_n\| \\ \le q^{k_n} \,\|x_n - P_{U\cap V} x_n\| \le q^{k_n} \,\|x_n\| \le q^{k_n} \,\|x\|,\end{gathered}$$

where we have used the fact that the sequence $\{\|x_n\|\}$ is decreasing.

Setting

$$\gamma_n = q^{k_n} \,\|x\|,$$

we obtain the last assumption of Proposition 3.1.

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