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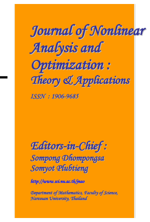

GENERALIZATIONS OF THE KKMF PRINCIPLE HAVING COERCING FAMILIES

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ABSTRACT. In 2005, Ben-El-Mechaiekh, Chebbi, and Florenzano [1] obtained a generalization of Ky Fan's 1984 KKM theorem on the intersection of a family of closed sets on non-compact convex sets in a topological vector space. They also extended the Fan-Browder fixed point theorem to multimaps on non-compact convex sets. In 2011, Chebbi, Gourdel, and Hammami [5] introduced a generalized coercivity type condition for multimaps defined on topological spaces endowed with a generalized convex structure and extended Fan's KKM theorem. In this paper, we show that better forms of theorems in [1, 3–5] can be deduced from a KKM theorem on abstract convex spaces in Park's sense [13–17].

KEYWORDS: KKM theorem; Fan's 1961 KKM lemma; 1984 KKM theorem; Fan-Browder fixed point theorem; Abstract convex space; (partial) KKM principle; (partial) KKM space.
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1. INTRODUCTION

One of the earliest equivalent formulations of the Brouwer fixed point theorem of 1912 is the theorem of Knaster, Kuratowski, and Mazurkiewicz (the KKM theorem for short) of 1929 [10] on the intersection of a family of closed sets. Actually, the KKM theorem was concerned with a particular type of multimaps, later called KKM maps by Dugundji and Granas [6]. The KKM theory (first called by the author in 1992; see [12, 15]) is the study of applications of various equivalent formulations of the KKM theorem and their generalizations.

From 1961 Ky Fan showed that the KKM theorem provides the foundation for many of the modern essential results in diverse areas of mathematical sciences. Actually, a milestone on the history of the KKM theory was erected by Fan in 1961 [7]. His 1961 KKM Lemma (or the Fan-KKM theorem or the KKMF principle [1]) extended the KKM theorem to arbitrary topological vector spaces and was applied to

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various problems in his subsequent papers. Moreover, his lemma was extended in 1979 and 1984 [8, 9] to the 1984 KKM theorem with a new coercivity (or compactness) condition for noncompact convex sets with new applications; see [12, 15].

In 2005, Ben-El-Mechaiekh, Chebbi, and Florenzano [1] obtained a generalization of the 1984 KKM theorem for KKM maps admitting a coercing family, and gave several deep examples of the family related to an exceptional family, an escaping sequence, an attracting trajectory, and others. They also extended the Fan-Browder fixed point theorem to multimaps on non-compact convex sets. Their generalizations of the KKM theorem is applied by Chebbi to some minimax inequality and equilibria in [3] and to some quasi-variational inequalities in [4]. Moreover, Chebbi, Gourdel, and Hammami in 2011 [5] introduced a generalized coercivity type condition for multimaps defined on topological spaces endowed with a generalized convex structure and Fan's KKM lemma.

Since 2006, the present author initiated the KKM theory on abstract convex spaces and obtained very general forms of KKM type theorems in [18, 19]. Moreover, in a recent paper [21], we introduced several generalizations of the 1984 KKM theorem and some known direct applications in order to reveal the close relationship among such generalizations. As a continuation, in the present paper, we show that some better forms of the KKM theorem, the fixed point theorem, and other results in [1, 3–5] can be deduced from a KKM theorem on abstract convex spaces in the sense of the author [13–23].

In Section 2 of this paper, we introduce the recent concepts of abstract convex spaces and partial KKM spaces. We also introduce one of the recent versions of general KKM type theorems in our previous works [18–23]. Section 3 is devoted to generalize the coercivity type conditions in [1] and [5]. In Sections 4 and 5, we show that better forms of main theorems in these two papers [1, 5] can be deduced from a KKM theorem on abstract convex spaces in the sense of [13–23]. Finally, Section 6 deals with improvements of results of [3, 4].

2. ABSTRACT CONVEX SPACES

Since 2006 we have introduced the concepts of abstract convex spaces, KKM spaces, and partial KKM spaces; see our recent works [17–22] and the references therein.

Definition 2.1. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D\rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$, where $\langle D\rangle$ is the set of all nonempty finite subsets of D.

For any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_A \mid A \in \langle D'\rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if, for any $N \in \langle D'\rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Definition 2.2. Let $(E, D; \Gamma)$ be an abstract convex space and Z a topological space. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y\in A} G(y) \qquad \text{for all } A \in \langle D\rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

A multimap $F : E \multimap Z$ is called a $\mathfrak{KC}$*-map* [resp., a $\mathfrak{KO}$*-map*] if, for any closed-valued [resp., open-valued] KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. In this case, we denote $F \in \mathfrak{KC}(E, D, Z)$ [resp., $F \in \mathfrak{KO}(E, D, Z)$].

Some remarks and examples on $\mathfrak{KC}$-maps and $\mathfrak{KO}$-maps can be seen in [13, 14]. In this paper, we need only the fact that any continuous function $s : E \longrightarrow Z$ belongs to $\mathfrak{KC}(E, D, Z)$.

Definition 2.3. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement $1_E \in \mathfrak{KC}(E, D, E)$; that is, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. The *KKM principle* is the statement $1_E \in \mathfrak{KC}(E, D, E) \cap \mathfrak{KO}(E, D, E)$; that is, the same property also holds for any open-valued KKM map.

An abstract convex space is called a (*partial*) *KKM space* if it satisfies the (partial) KKM principle, respectively.

Example 2.4. The following are typical examples of KKM spaces. Others can be seen in [15–17] and the references therein.

(1) A *convex space* $(X, D) = (X, D; \Gamma)$ is a triple where X is a subset of a vector space, $D \subset X$ such that $\operatorname{co} D \subset X$, and each Γ_A is the convex hull of $A \in \langle D\rangle$ equipped with the Euclidean topology. This concept generalizes the one due to Lassonde [11] for $X = D$.

(2) A *generalized convex space* or a G*-convex space* $(X, D; \Gamma)$ is an abstract convex space such that for each $A \in \langle D\rangle$ with the cardinality $|A| = n+1$, there exists a continuous function $\phi_A : \Delta_n \longrightarrow \Gamma(A)$ such that $J \in \langle A\rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, Δ_n is the standard n-simplex with vertices $\{e_i\}_{i=0}^n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A\rangle$.

When $X = D$, a G-convex space is called an L-space; see [5].

(3) A *space having a family* $\{\phi_A\}_{A\in\langle D\rangle}$ or simply a ϕ_A*-space*

$$(X, D; \{\phi_A\}_{A\in\langle D\rangle})$$

consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \longrightarrow X$ (that is, singular n-simplices) for $A \in \langle D\rangle$ with the cardinality $|A| = n + 1$.

A subset C of X is said to be ϕ_A*-convex* with respect to a subset $D' \subset D$ if for each $B \in \langle D'\rangle$, we have $\operatorname{Im}\phi_B := \phi_B(\Delta_{|B|-1}) \subset C$.

For a ϕ_A-space $(X, D; \{\phi_A\})$, the corresponding abstract convex space $(X, D; \Gamma)$ with $\Gamma_A := \phi_A(\Delta_n)$ for $A \in \langle D\rangle$ with $|A| = n + 1$ is a KKM space. This KKM space may not be G-convex.

We have the following diagram for triples $(E, D; \Gamma)$:

Simplex $\Longrightarrow$ Convex subset of a t.v.s. $\Longrightarrow$ Convex space $\Longrightarrow$ H-space
$\Longrightarrow$ G-convex space $\Longrightarrow$ ϕ_A-space $\Longrightarrow$ KKM space
$\Longrightarrow$ Partial KKM space $\Longrightarrow$ Abstract convex space.

Consider the following related four conditions for a multimap $G : D \multimap Z$ from a set D into a topological space Z:

(a) $\bigcap_{y\in D} \overline{G(y)} \neq \emptyset$ implies $\bigcap_{y\in D} G(y) \neq \emptyset$.

(b) $\bigcap_{y\in D}\overline{G(y)} = \overline{\bigcap_{y\in D} G(y)}$ (G is *intersectionally closed-valued*).

(c) $\bigcap_{y\in D}\overline{G(y)} = \bigcap_{y\in D} G(y)$ (G is *transfer closed-valued*).

(d) G is closed-valued.

From the definition of $\mathfrak{KC}$-maps, we have a whole intersection property of the Fan type under certain "coercivity" conditions. The following is given in [18, 19, 21–23]:

Theorem 2.5. *Let $(E, D; \Gamma)$ be an abstract convex space, Z a topological space, $F \in \mathfrak{KC}(E, D, Z)$, and $G : D \multimap Z$ a map such that*
(1) *$\overline{G}$ is a KKM map w.r.t. F; and*
(2) *there exists a nonempty compact subset K of Z such that either*
(i) *$K \supset \bigcap\{\overline{G(y)} \mid y \in M\}$ for some $M \in \langle D\rangle$; or*
(ii) *for each $N \in \langle D\rangle$, there exists a Γ-convex subset L_N of E relative to some $D' \subset D$ such that $N \subset D'$, $\overline{F(L_N)}$ is compact, and*

$$K \supset \overline{F(L_N)} \cap \bigcap\{\overline{G(y)} \mid y \in D'\}.$$

Then we have

$$\overline{F(E)} \cap K \cap \bigcap\{\overline{G(y)} \mid y \in D\} \neq \emptyset.$$

Furthermore,
(α) *if G is transfer closed-valued, then $\overline{F(E)} \cap K \cap \bigcap\{G(y) \mid y \in D\} \neq \emptyset$; and*
(β) *if G is intersectionally closed-valued, then $\bigcap\{G(y) \mid y \in D\} \neq \emptyset$.*

Remark 2.6. 1. Taking $\overline{K}$ instead of K, we may assume K is closed and the closure notations in (i) and (ii) can be erased.

2. In a recent work [23], we showed that a particular form of Theorem 2.5 for $F = 1_E$ unifies several important KKM type theorems appeared in history.

3. GENERALIZATIONS OF VARIOUS COERCING FAMILIES

In this section, we obtain generalizations of coercivity conditions considered in [1] and [5].

Let us begin with the following particular form of the condition (ii) in Theorem 2.5 with $sG : D \multimap Z$ instead of $G : D \multimap Z$:

(I) Let $(E, D; \Gamma)$ be an abstract convex space, $G : D \multimap E$ a multimap, Z a topological space, and $s : E \longrightarrow Z$ a continuous map such that

(C) there exists a nonempty compact subset K of Z such that, for each $N \in \langle D\rangle$, there exists a compact Γ-convex subset L_N of E relative to some $D' \subset D$ such that $N \subset D'$ and

$$s(L_N) \cap \bigcap_{y\in D'} sG(y) \subset K.$$

Note that $s \in \mathfrak{KC}(E, D, Z)$.

Under the situation of (I), we have the following:

Proposition 3.1. *If $(E, D; \Gamma)$ is a partial KKM space, then so is the abstract convex space $(Z, D; s\Gamma)$, where $s\Gamma : \langle D\rangle \multimap Z$.*

Proof. Let $G' : D \multimap Z$ be a closed-valued KKM map, that is, for any $A \in \langle D\rangle$, $s\Gamma(A) \subset G'(A)$ or $\Gamma(A) \subset (s^{-1}G')(A)$. Then $s^{-1}G' : D \multimap E$ is a closed-valued KKM map on the partial KKM space $(E, D; \Gamma)$. Hence $\{s^{-1}G'(a)\}_{a\in D}$ has the finite intersection property and so does the family $\{G'(a)\}_{a\in D}$. □

Note that the proof of Proposition 3.1 also hods for open-valued KKM maps and for KKM spaces.

Proposition 3.2. *The set* $s(L_N)$ *is a compact* $s\Gamma$*-convex subset of* $(Z, D; s\Gamma)$.

Proof. Since L_N is compact and s is continuous, $s(L_N)$ is compact. Since L_N is Γ-convex relative to some $D' \subset D$ such that $N \subset D'$, for any $A \in \langle D' \rangle$, we have $\Gamma(A) \subset L_N$ and hence $s\Gamma(N) \subset s(L_N)$. Therefore, $s(L_N)$ is $s\Gamma$-convex relative to $D' \subset D$. □

In 2011, Chebbi et al. [5] introduced the notion of coercing family in L-spaces for a given map as follows:

(II) Let D be an arbitrary set in an L-space (E, Γ), Z a topological space, and $s : E \longrightarrow Z$ a continuous map. A family $\{(C_a, K)\}_{a \in E}$ is said to be L-coercing for a map $F : D \multimap Z$ with respect to s if

(i) K is a compact subset of Z;

(ii) for each $N \in \langle D \rangle$, there exists a compact L-convex set L_N in E containing N such that

$$x \in L_N \;\Rightarrow\; C_x \cap D \subset L_N \cap D;$$

(iii) $\{x \in E \mid s(x) \in \bigcap_{y \in C_x \cap Z} F(y)\} \subset s^{-1}(K)$.

Proposition 3.3. *Definition* (II) *implies* (I).

Proof. Under the situation of (II), note that $(E, D; \Gamma)$ is a G-convex space and hence a (partial) KKM space. Let $G := s^{-1}F : D \multimap E$ and, for any $N \in \langle D \rangle$, we have a compact Γ-convex subset L_N of E containing N. Choose an $x \in L_N$ and let $D' \equiv (C_x \cap D) \cup N \subset L_N \cap D$ by (ii) (and Remark 1 in [3]). Then L_N is Γ-convex relative to $D' \subset D$ containing N. Moreover, by (iii),

$$x \in \bigcap_{y \in D'} G(y) = \bigcap_{y \in C_x \cap D} s^{-1}F(y) \subset s^{-1}(K).$$

Hence

$$s(x) \in s(L_N) \cap \bigcap_{y \in D'} sG(y) \subset K.$$

Therefore (I) holds. □

Motivated by [1], we define the following:

(III) Let $(E, D; \Gamma)$ be an abstract convex space and Z a topological space. We say that a map $G : D \multimap Z$ has a *coercing family* $\{(D_i, K_i)\}_{i \in I}$ if and only if

(1) for each $i \in I$, K_i is a compact subset of Z and $D_i \subset D$ such that, for each $N \in \langle D \rangle$, there exist a compact subset L_N^i of E that is Γ-convex relative to $D_i \cup N$;

(2) for each $i \in I$, there exists $k \in I$ with $\bigcap_{y \in D_k} F(y) \subset K_i$.

This definition improves the following coercivity in the sense of Ben-El-Mechaiekh, Chebbi, and Florenzano in [1];

(IV) [1] Consider a subset X of a Hausdorff topological vector space and a topological space Z. A family $\{(D_i, K_i)\}_{i \in I}$ of pairs of sets is said to be *coercing* for a map $F : X \multimap Z$ if and only if:

(i) for each $i \in I$, D_i is contained in a compact convex subset of X, and K_i is a compact subset of Z;

(ii) for each $i, j \in I$, there exists $k \in I$ such that $D_i \cup D_j \subset D_k$;

(iii) for each $i \in I$, there exists $k \in I$ with $\bigcap_{x \in D_k} F(x) \subset K_i$.

If I is a singleton, the family is called a *single* coercing family.

Remark 3.4. In [1], it is noted that the condition (iii) holds *if and only if* the "dual" map $\Phi : Z \multimap X$ of F, defined by $\Phi(z) = X \setminus F^-(z),\ z \in Z$ verifies
(iii)$'$ $\forall i \in I,\ \exists k \in I,\ \forall z \in Z \setminus K_i,\quad \Phi(z) \cap C_k \neq \emptyset.$

In [1], there are given several deep examples of condition (iii)$'$ related to an exceptional family, an escaping sequence, an attracting trajectory, and others.

Remark 3.5. In [5], it is shown that L-coercing families in (II) contain coercing families in the sense of (IV).

Here we show that (III) is equivalent to a particular case of the coercivity (I) for abstract convex spaces:

Proposition 3.6. *Let $(E, D; \Gamma)$ be an abstract convex space. A map $G : D \multimap E$ admits a coercing family in the sense of* (III) *if and only if the coercivity* (I) *with $E = Z$ and $s = 1_E$ holds.*

Proof. When I is a singleton, then the existence of a coercing family implies the coercivity (I) with $E = Z$ and $s = 1_E$.

Conversely, choose an arbitrary $i \in I$ and let $K := K_i$. For an $N \in \langle D \rangle$, let $D' := D_k \cup N$ with D_k in (III)(2). Since there exists a compact Γ-convex subset $L_N := L_N^k$ of E relative to D', by (III)(2) again, we have

$$L_N \cap \bigcap_{y \in D'} F(y) \subset L_N \cap \bigcap_{y \in D_k} G(y) \subset K.$$

Therefore, the coercivity condition (I) with $E = D$ and $s = 1_E$ holds. □

Note that all of (I)-(IV) are examples of the coercivity (ii) in Theorem 2.5.

4. GENERALIZATIONS OF THE KKM THEOREM

In this section, we show that better forms of KKM theorems in [1] and [5] can be deduced from the KKM theorem 2.5 on abstract convex spaces.

Theorem 4.1. *Let $(E, D; \Gamma)$ be an abstract convex space, Z an arbitrary topological space and $G : D \multimap Z$ a closed-valued multimap. Suppose that there exists a continuous map $s : E \longrightarrow Z$ such that:*
(1) *the multimap $R : D \multimap E$ defined by $R(y) := s^{-1}(G(y))$ is KKM;*
(2) *the coercivity condition* (I) *holds for R instead of G.*
Then we have $K \cap \bigcap_{y \in D} G(y) \neq \emptyset$.

Proof. We apply Theorem 2.5 with $F := s$.

(1) Since $s^{-1}G$ is a closed-valued KKM map by (1), $\Gamma_A \subset R(A) = s^{-1}G(A)$ and $s\Gamma_A \subset sR(A) = G(A)$ for all $A \in \langle D \rangle$. Therefore $\overline{G}$ is a KKM map w.r.t. s.

(2) Condition (2) implies (ii) in Theorem 2.5 with $F := s$ and $G := sR$.

Therefore, by the case (ii) of Theorem 2.5, we have

$$s(E) \cap K \cap \bigcap_{y \in D} sR(y) \neq \emptyset.$$

This implies the conclusion. □

The main theorem of [5] is the particular case of Theorem 4.1 under the assumption of (II) as follows:

Corollary 4.2. *[5] Let D be an arbitrary set in the L-space (E, Γ), Z an arbitrary topological space and $F : D \multimap Z$ a map with quasi-compactly closed values. Suppose that there exists a continuous function $s : E \longrightarrow Z$ such that:*

(1) *the map $R : D \multimap E$ defined by $R(y) = s^{-1}(F(y))$ is KKM;*

(2) *there exists an L-coercing family for F with respect to s as in* (II).

Then $K \cap \bigcap_{x \in D} F(x) \neq \emptyset$.

The main theorem of [1] is the particular case $s = 1_Y$ of Theorem 4.1 under the assumption of (IV) as follows:

Corollary 4.3. *[1] Let E be a Hausdorff topological vector space, Y a convex subset of E, X a non-empty subset of Y, and $F : X \multimap Y$ a KKM map with compactly closed (in Y) values. If F admits a coercing family in the sense of* (IV)*, then $\bigcap_{x \in X} F(x) \neq \emptyset$.*

Corollary 4.4. *From the above corollaries, we notice the following:*

1. The quasi-compactly closed sets are compactly closed sets in modern usage and can be replaced by mere closed sets by adopting compactly generated extension of the original topology.

2. Our proofs are based on Theorem 2.5 and different from that of [1] and [3].

3. In view of (III), condition (ii) in (IV) of a coercing family is redundant.

4. The existence of a coercing family (IV) is simply equivalent to that of a single coercing family.

5. In Corollaries 4.2 and 4.3, F can be transfer closed-valued or intersectionally closed-valued.

6. As was noted in [1], if the coercing family is single, then Corollary 4.3 reduces to the 1984 KKM theorem 4 of Fan [9] which in turn generalizes the KKMF principle.

5. GENERALIZATIONS OF THE FAN-BROWDER FIXED POINT THEOREMS

In this section, we show that better forms of the Fan-Browder fixed point theorems in [1] and [5] can be deduced from the KKM theorem 2.5 on abstract convex spaces.

From Theorem 2.5, we also obtain the following Fan-Browder type alternative:

Theorem 5.1. *Let $(E, D; \Gamma)$ be a partial KKM space, and $S : E \multimap D$, $T : E \multimap E$ maps satisfying*

(1) *$S^-(y)$ is open for each $y \in D$;*

(2) *$T(x) \supset \mathrm{co}_\Gamma S(x)$ for each $x \in E$.*

Suppose that there exists a nonempty compact subset K of E satisfying

(3) *for each $N \in \langle D \rangle$, there exist $D' \subset D$ containing N and a compact Γ-convex subset L_N of E relative to D' such that*

$$L_N \cap \bigcap \{E \setminus S^-(y) \mid y \in D'\} \subset K.$$

Then either (a) *S has a maximal element $x_0 \in K$, that is, $S(x_0) = \emptyset$; or* (b) *T has a fixed point $x_1 \in E$, that is, $x_1 \in T(x_1)$.*

Proof. Suppose T has no fixed point. Define a map $G : D \multimap E$ by

$$G(y) := E \setminus S^-(y) = \{x \in E \mid y \notin S(x)\}, \quad y \in D.$$

Then G is closed-valued. Moreover, G is a KKM map.

In fact, suppose on the contrary that there exists an $N \in \langle D \rangle$ such that $\Gamma_N \not\subset G(N)$; that is, there exists an $x \in \Gamma_N$ such that $x \notin G(y)$ for all $y \in N$. In other words, $N \in \langle D \setminus G^-(x) \rangle$ and

$$y \notin G^-(x) \Leftrightarrow x \notin G(y) = E \setminus S^-(y) \Leftrightarrow x \in S^-(y) \Leftrightarrow y \in S(x)$$

for all $y \in N$. Hence $N \subset S(x)$ and, by (2), we have $x \in \Gamma_N \subset T(x)$. This is a contradiction.

Note that (3) implies condition (ii) of Theorem 2.5 with $E = D$ and $s = 1_E \in \mathfrak{KC}(E, D, E)$ since $(E, D; \Gamma)$ is a partial KKM space. Therefore, by Theorem 2.5, we have $K \cap \bigcap_{y \in D} G(y) \neq \emptyset$.

Then we have an $x_0 \in K$ and $x_0 \in G(y) = E \setminus S^-(y)$ or $y \notin S(x_0)$ for all $y \in D$. Hence S has a maximal element $x_0 \in K$. □

From Theorem 5.1, we obtain the following fixed point result [1]:

Corollary 5.2. *[1] Let X be a non-empty convex subset of a Hausdorff topological vector space and let $\Phi : X \multimap X$ be a map with open fibers (in X) and non-empty values. If Φ admits a single coercing family in the sense of* (IV) *satisfying* (iii)$'$, *then the map* co(Φ) *has a fixed point.*

Proof. We will use Theorem 5.1 with $E = D = X$, $\Gamma = \mathrm{co}$, $S = \Phi$, $T = \mathrm{co}(\Phi)$. Since Φ has non-empty values, it does not have a maximal element. Now it suffices to show that (iii)$'$ implies condition (3) of Theorem 5.1.

Suppose K is a compact subset of X and C is contained in a compact convex subset L of X. Let $N \in \langle X \rangle$. Since X is a convex subset of a Hausdorff topological vector space, there exists a compact convex subset L_N of X containing $D' := L \cup N$. Note that $\Phi(x) \cap D' \supset \Phi(x) \cap C \neq \emptyset$ for all $x \in X \setminus K$ by (iii)$'$, that is,

$$x \in X \setminus K \Rightarrow \Phi(x) \cap D' \neq \emptyset \Rightarrow \exists y \in \Phi(x) \cap D' \Rightarrow x \in \Phi^-(y),\ \exists y \in D'.$$

Now we have

$$x \in L_N \cap \bigcap_{y \in D'} (X \setminus \Phi^-(y)) \Rightarrow x \in X \setminus \Phi^-(y),\ \forall y \in D' \Rightarrow x \notin \Phi^-(y),\ \forall y \in D'.$$

Therefore, $x \notin X \setminus K$ and hence $x \in K$. So condition (3) of Theorem 5.1 holds. □

Now we can obtain an equivalent variant of Theorem 5.1:

Corollary 5.3. *Let $(E, D; \Gamma)$ be a partial KKM space, Z an arbitrary topological space and $G : D \multimap Z$, $H : E \multimap Z$ multimaps. Suppose that there exists a continuous map $s : E \longrightarrow Z$ such that:*

(1) *$G(y)$ is open for each $y \in D$;*

(2) *$s^{-1}H(x) \supset \mathrm{co}_\Gamma G^- s(x)$ for each $x \in E$.*

Suppose that there exists a nonempty compact subset K of E satisfying

(3) *for each $N \in \langle D \rangle$, there exist $D' \subset D$ containing N and a compact Γ-convex subset L_N of E relative to D' such that*

$$s(L_N) \cap \bigcap \{Z \setminus s^{-1}G(y) \mid y \in D'\} \subset K.$$

Then either (a) *$G^- s$ has a maximal element $x_0 \in E$, that is, $G^- s(x_0) = \emptyset$; or* (b) *$s^{-1}H$ has a fixed point $x_1 \in E$, that is, $s(x_1) \in H(x_1)$.*

Proof. In view of Propositions 3.1 and 3.2, $(Z, D; s\Gamma)$ is a partial KKM space and $s(L_N)$ is a compact $s\Gamma$-convex subset relative to D'. We apply Theorem **??** replacing $(E, D; \Gamma)$ by $(Z, D; s\Gamma)$, $S := G^- s$ and $T := s^{-1}H$. Then

(1) $S^- = s^{-1}G$ is open-valued since so is G and s is continuous.

(2) $T(x) = s^{-1}H(x) \supset \text{co}_\Gamma G^- s(x) = \text{co}_\Gamma S(x)$ for each $x \in E$.
(3) Condition (3) of Theorem 5.1 with $S^- = s^{-1}G$ holds.
Therefore, by Theorem 5.1, the conclusion follows. □

Remark 5.4. Note that we deduced Corollary 5.3 from Theorem 5.1. Conversely, Corollary 5.3 for $E = Z$ and $s = 1_E$ reduces to Theorem 5.1.

The following is Theorem 2 in [5]:

Corollary 5.5. *[5] Let $(X;\Gamma)$ be an L-space, Z an arbitrary topological space, $s : X \longrightarrow Z$ a continuous map and $S : X \multimap Z$ a multimap such that:*
(i) *for each $x \in X$, $S(x)$ is quasi-compactly open in Z;*
(ii) *for each $z \in Z$, $S^{-1}(z)$ is nonempty and L-convex;*
(iii) *there exists an L-coercing family $\{(C_x, K)\}_{x\in X}$ for the map $Q(x) = Z \setminus S(x)$ with respect to s.*

Then there exists $x_0 \in X$ such that $s(x_0) \in S(x_0)$. In particular, if s is the identity map, then S has a fixed point.

Proof. Put $E = D = X$ and $G = H = S$ in Corollary 5.3. □

As was noted in [1], Theorems in Sections 4 and 5 can be used to extend existing results on various equilibrium problems, solvability of complementarity problems, existence of zero on non-compact domains, and existence of equilibria for qualitative games and abstract economies.

6. COMMENTS ON GENERAL MINIMAX INEQUALITIES AND APPLICATIONS

It is well-known that any KKM type theorem can be reformulated equivalently to the Fan-Browder type fixed point theorems, matching theorems, minimax inequalities, and so on.

In this section, we indicate that results of Chebbi in [3, 4] can be improved following our preceding arguments.

The following is a KKM type minimax inequality given in Theorem 5.3 in [22]

Theorem 6.1. *[22] Let $(E, D;\Gamma)$ be a partial KKM space. Let $f : E \times D \longrightarrow \overline{\mathbb{R}}$ be an extended real-valued function and $\gamma \in \overline{\mathbb{R}}$ such that*
(1) *for each $y \in D$, $\{x \in E \mid f(x, y) \le \gamma\}$ is intersectionally closed [resp., transfer closed];*
(2) *for each $N \in \langle D\rangle$ and $x \in \Gamma_N$, $\min\{f(x, y) \mid y \in N\} \le \gamma$; and*
Suppose that there exists a nonempty compact subset K of E satisfying
(3) *for each $N \in \langle D\rangle$, there exist $D' \subset D$ containing N and a compact Γ-convex subset L_N of E relative to D' such that*

$$L_N \cap \bigcap_{y\in D'} \{x \in E \mid f(x, y) \le \gamma\} \subset K.$$

Then (a) *there exists a $\hat{x} \in E$ [resp., $\hat{x} \in K$] such that*

$$f(\hat{x}, y) \le \gamma \quad \text{for all } y \in D; \text{ and}$$

(b) *if $E = D$ and $\gamma = \sup_{x\in E} f(x, x)$, then we have the minimax inequality:*

$$\inf_{y\in E} \sup_{x\in E} f(x, y) \le \sup_{x\in E} f(x, x).$$

Corollary 6.2. *[3] Let X be a nonempty convex subset of a t.v.s. E, and $f : X \times X \longrightarrow \overline{\mathbb{R}}$ be a function satisfying*

(i) *f is l.s.c. in the first variable on each compact convex subsets of X;*
(ii) *for each $A \in \langle X \rangle$, $\sup_{x\in \text{co}\, A} \min_{y\in A} f(x, y) \leq 0$; and*
(iii) *the coercivity condition* (IV) *with $X = Z$ and*

$$F(y) := \{x \in X \mid f(x, y) \leq 0\} \text{ for } y \in X.$$

Then there exists an $x_0 \in X$ such that $f(x_0, y) \leq 0$ for all $y \in X$.

Proof. Note that X can be regarded a convex space in the sense of Lassonde [11] and endowed the compactly generated extension of of its original topology. Then (i) becomes simply "f is l.s.c." and hence, condition (1) of Theorem 6.1 is satisfied. Moreover, it is clear that (ii) implies (2) of Theorem 6.1. Further, (iii) implies the coercivity condition (I) in Section 3 with $s = 1_E$ and $G(y) := \{x \in E \mid f(x, y) \leq \gamma\}$ for $y \in D$. Therefore, the conclusion of Corollary 5.2 follows from Theorem 6.1(a) with $\gamma = 0$. □

Corollary 6.2 is applied to some equilibrium problems in [3] and to some quasi-variational inequalities in [4]. Note that Corollary 6.2 can be improved by adopting more general conditions (I)–(III) with $s = 1_E$ and $Z = E$. Moreover, any interested reader can check that all results in [3] and [4] can be improved by applying Theorem 6.1 instead of Corollary 6.2.

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