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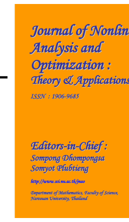

ON THE MEANS OF PROJECTIONS ON CAT(0) SPACES

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ABSTRACT. We improve a result on approximation a common element of two closed convex subsets of a complete CAT(0) space appeared as Theorem 4.1 in [2]. New practical iterative scheme is presented and conditions on two given sets are relaxed.

KEYWORDS: Projection; CAT(0) space.
AMS Subject Classification: 47H09 47H10

1. INTRODUCTION

von Neumann introduced the alternating projection method and proved the following strong convergence in Hilbert spaces [cf. 2]:

Theorem 1.1 (von Neumann)**.** *Let H be a Hilbert space and $A, B \subset H$ its closed subspaces. Assume $x_0 \in H$ is a starting point and $\{x_n\} \subset H$ the sequence generated by*

$$x_{2n-1} = P_A(x_{2n-2}), \quad x_{2n} = P_B(x_{2n-1}), \ n \in \mathbb{N}, \tag{1.1}$$

where P_A, P_B are projection mappings from H to A and B respectively. Then $\{x_n\}$ converges in norm to a point from $A \cap B$.

When "subspaces" are replaced by "convex subsets", we only have "weak convergence" for the alternating projections:

Theorem 1.2. [3] *Let H be a Hilbert space and $A, B \subset H$ closed convex sets with $A \cap B \neq \emptyset$. Assume $x_0 \in H$ is a starting point and $\{x_n\} \subset H$ the sequence generated by (1.1). Then $\{x_n\}$ weakly converges to a point from $A \cap B$.*

It took 39 years since 1965 until Hundal [7] in 2004 could provide a counter example:

Example 1.3. [7] *There exist a hyperplane $A \subset \ell_2$, a convex cone $B \subset \ell_2$ and a point $x_0 \in \ell_2$ such that the sequence generated by (1.1) from the starting point x_0 converges weakly to a point in $A \cap B$ but not in norm.*

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In 2011, Bačák, Searston, Sims [2] extend the result of Bregman for CAT(0) spaces.

Theorem 1.4. [2, Theorem 4.1] *Let X be a complete CAT(0) space and $A, B \subset X$ convex closed subsets such that $A \cap B \neq \emptyset$. Let $x_0 \in X$ be a starting point and $\{x_n\} \subset X$ be the sequence generated by (1.1). Then:*

(i) *$\{x_n\}$ weakly converges to a point $x \in A \cap B$.*
(ii) *If A and B are boundedly regular, then $x_n \longrightarrow x$.*
(iii) *If A and B are boundedly linearly regular, then $x_n \longrightarrow x$ linearly.*
(iv) *If A and B are linearly regular, then $x_n \longrightarrow x$ linearly with a rate independent of the starting point.*

It is the aim of this paper to present an iterative sequence which strongly converges to a common point of the sets A and B. We do not impose any requirements on A and B as stated in (ii).

2. PRELIMINARIES

Let (X, d) be a metric space. A *geodesic* joining $x \in X$ to $y \in X$ is a mapping c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. Obviously, c is an isometry and $d(x, y) = l$. We call the image of c a *geodesic segment* joining x and y. If it is unique this geodesic is denoted $[x, y]$. Write $c(\alpha 0 + (1 - \alpha)l) = \alpha x \oplus (1 - \alpha)y$ for $\alpha \in (0, 1)$. We also write the midpoint $\frac{1}{2}x \oplus \frac{1}{2}y$ of a segment $[x, y]$ as $\frac{x \oplus y}{2}$. The space X is said to be a *geodesic space* if every two points of X are joined by a geodesic. It is said to be of *hyperbolic type* [6] if it satisfies the following inequality:

$$d(p, \alpha x \oplus (1 - \alpha)y) \leq \alpha d(p, x) + (1 - \alpha)d(p, y) \tag{2.1}$$

for all $p \in X$. Following [5], let $\{v_1, v_2, ..., v_n\} \subset X$ and $\{\lambda_1, \lambda_2, ..., \lambda_n\} \subset (0, 1)$ with $\sum_{i=1}^n \lambda_i = 1$ and write, by induction,

$$\bigoplus_{i=1}^n \lambda_i v_i := (1 - \lambda_n)\Big(\frac{\lambda_1}{1 - \lambda_n} v_1 \oplus \frac{\lambda_2}{1 - \lambda_n} v_2 \oplus \cdots \oplus \frac{\lambda_{n-1}}{1 - \lambda_n} v_{n-1}\Big) \oplus \lambda_n v_n. \tag{2.2}$$

Note for an example that $\frac{1}{3}v_1 \oplus \frac{1}{3}v_2 \oplus \frac{1}{3}v_3$ and $\frac{1}{3}v_2 \oplus \frac{1}{3}v_1 \oplus \frac{1}{3}v_3$ are not necessary coincide. Under (2.1) we can see that

$$d\left(\bigoplus_{i=1}^n \lambda_i v_i, x\right) \leq \sum_{n=1}^n \lambda_i d(v_i, x) \tag{2.3}$$

for each $x \in X$.

A metric space X is said to be a *CAT(0) space* (cf.[4] p.163) if it is a geodesic space satisfying one of the following equivalent conditions.

(i) **(CN) inequality**: If $x_0, x_1 \in X$, then

$$d^2\left(y, \frac{x_0 \oplus x_1}{2}\right) \leq \frac{1}{2}d^2(y, x_0) + \frac{1}{2}d^2(y, x_1) - \frac{1}{4}d^2(x_0, x_1), \text{ for all } y \in X.$$

(ii) **Law of cosine**: If $a = d(p, q), b = d(p, r), c = d(q, r)$ and ξ is the Alexandrov angle at p between $[p, q]$ and $[p, r]$, then $c^2 \geq a^2 + b^2 - 2ab\cos\xi$.

Lemma 2.1. [4, Proposition 2.2] *Let X be a CAT(0) space. Then for each $p, q, r, s \in X$ and $\alpha \in [0, 1]$,*

$$d(\alpha p \oplus (1 - \alpha)q, \alpha r \oplus (1 - \alpha)s) \leq \alpha d(p, r) + (1 - \alpha)d(q, s). \tag{2.4}$$

In particular, (2.1) holds in CAT(0) spaces.

Let C be a nonempty subset of X. We will denote the family of nonempty bounded closed subsets of C by $BC(C)$ and the family of nonempty compact subsets of C by $K(C)$. Let $H(\cdot,\cdot)$ be the *Hausdorff distance* on $BC(X)$, that is,

$$H(A,B) = \max\left\{\sup_{a\in A} dist(a,B), \sup_{b\in B} dist(b,A)\right\}, \quad A,B\in BC(X),$$

where $dist(a,B) = \inf\{d(a,b) : b\in B\}$ is the distance from the point a to the subset B.

A mapping $t : C \longrightarrow C$ and a multivalued mapping $T : C \longrightarrow BC(C)$ are said to be *nonexpansive* if for each $x,y\in C$,

$$d(tx,ty) \le d(x,y), \text{ and}$$
$$H(Tx,Ty) \le d(x,y),$$

respectively. If $tx = x$, we call x a fixed point of a single valued mapping t. And if $x \in Tx$, we call x a fixed point of a multivalued mapping T. We use the notation $Fix(S)$ to stand for the set of all fixed points of a mapping S. Thus $Fix(t)\cap Fix(T)$ is the set of common fixed points of t and T, i.e., $x \in Fix(t)\cap Fix(T)$ if and only if $x = tx \in Tx$.

Let $\{\lambda_n\}$ be a given sequence in $(0,1)$ such that $\sum_{n=1}^{\infty}\lambda_n = 1$, let $\{v_n\}$ be a bounded sequence in X and let v_0 be an arbitrary point in X. Let $\lambda'_n = \sum_{i=n+1}^{\infty}\lambda_i$ and assume that $\sum_{i=n}^{\infty}\lambda'_i \longrightarrow 0$ as $n \longrightarrow \infty$. In [5] the element $\bigoplus_{n=1}^{\infty}\lambda_n v_n$ has been defined. Here is its description. Set

$$s_n := \lambda_1 v_1 \oplus \lambda_2 v_2 \oplus \cdots \oplus \lambda_n v_n \oplus \lambda'_n v_0.$$

Thus, by (2.2),

$$s_n = \Big(\sum_{i=1}^{n}\lambda_i\Big) w_n \oplus \lambda'_n v_0, \tag{2.5}$$

where $w_1 = v_1$ and for each $n \ge 2$,

$$w_n = \frac{\lambda_1}{\sum_{i=1}^{n}\lambda_i} v_1 \oplus \frac{\lambda_2}{\sum_{i=1}^{n}\lambda_i} v_2 \oplus \cdots \oplus \frac{\lambda_n}{\sum_{i=1}^{n}\lambda_i} v_n.$$

We know that $\{s_n\}$ is a Cauchy sequence. Thus $s_n \longrightarrow x$ as $n \longrightarrow \infty$ for some $x\in X$. Write

$$x = \bigoplus_{n=1}^{\infty}\lambda_n v_n.$$

By (2.5), $d(s_n,w_n) \le \lambda'_n d(w_n,v_0)$, it is seen that $\lim_{n\longrightarrow\infty} s_n = \lim_{n\longrightarrow\infty} w_n$. Thus the limit x is independent of the choice of v_0. Moreover, it had been shown in [5] that

(A): if y_0 and v_n belong to X, $d(v_n,y_0) = d(x,y_0)$ for all n where $x = \bigoplus_{n=1}^{\infty}\lambda_n v_n$, then $v_n = x$ for all n.

Lemma 2.2. [5, Lemma 3.8] *Let C be a nonempty closed convex subset of a complete CAT(0) space X, let $\{t_n : n\in\mathbb{N}\}$ be a family of single-valued nonexpansive mappings on C. Suppose $\bigcap_{n=1}^{\infty} Fix(t_n)$ is nonempty. Define $t : C \longrightarrow C$ by*

$$t(x) = \bigoplus_{n=1}^{\infty}\lambda_n t_n(x)$$

for all $x\in C$ where $\{\lambda_n\}\subset(0,1)$ with $\sum_{n=1}^{\infty}\lambda_n = 1$ and $\sum_{i=n}^{\infty}\lambda'_i \longrightarrow 0$ as $n \longrightarrow \infty$. Then t is nonexpansive and $Fix(t) = \bigcap_{n=1}^{\infty} Fix(t_n)$.

Theorem 2.3. [8, Lemma 2.2] *Let C be a nonempty closed convex subset of a complete CAT(0) space X, let $t : C \to C$ be nonexpansive, fix $u \in C$, and for each $s \in (0, 1)$ let x_s be the point of $[u, t(x_s)]$ satisfying*

$$d(u, x_s) = sd(u, t(x_s)).$$

Then $Fix(t) \neq \emptyset$ if and only if $\{x_s\}$ remains bounded as $s \longrightarrow 1$. In this case, the following statements hold:

(1) *$\{x_s\}$ converges to the unique fixed point z of t which is nearest to u;*
(2) *$d^2(u, z) \leq \mu_n d^2(u, u_n)$ for all Banach limits μ and all bounded sequences $\{u_n\}$ with $d(u_n, t(u_n)) \longrightarrow 0$.*

We will follow the proof of the following theorem to prove our main result (Theorem 3.1).

Theorem 2.4. [5, Theorem 3.7] *Let C be a nonempty closed convex subset of a complete CAT(0) space X. Let $\{t_n : C \longrightarrow C\}$ be a countable family of nonexpansive mappings and $T : C \longrightarrow K(C)$ be a nonexpansive mapping with $\bigcap_{n=1}^{\infty} Fix(t_n) \cap Fix(T) \neq \emptyset$. Suppose that $T(p) = \{p\}$ for all $p \in \bigcap_{n=1}^{\infty} Fix(t_n) \cap Fix(T)$. Let t and $\{\lambda_n\}$ be as in Lemma 2.2. Suppose that $u, z_1 \in C$ are arbitrarily chosen and $\{z_n\}$ is defined by*

$$z_{n+1} = \alpha_n u \oplus (1 - \alpha_n) \left(\frac{1}{2} w_n(z_n) \oplus \frac{1}{2} y_n \right), \quad n \in \mathbb{N}, \tag{2.6}$$

such that $d(y_n, y_{n+1}) \leq d(z_n, z_{n+1})$ for all $n \in \mathbb{N}$, where $y_n \in T(z_n)$ and $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying

(C1) $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$ *or* $\lim_{n\longrightarrow\infty}(\alpha_n/\alpha_{n+1}) = 1$.

Then $\{z_n\}$ converges to the unique point of $\bigcap_{n=1}^{\infty} Fix(t_n) \cap Fix(T)$ which is nearest to u.

In the course of the proof of Theorem 2.4, the following results play important role.

Lemma 2.5. [9, Proposition 2] *Let a be a real number and let $(a_1, a_2, ...) \in \ell^\infty$ be such that $\mu_n(a_n) \leq a$ for all Banach limits μ and $\limsup_n (a_{n+1} - a_n) \leq 0$. Then $\limsup_n a_n \leq a$.*

Lemma 2.6. [1, Lemma 2.3] *Let $\{s_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ a sequence of real numbers in $[0, 1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\eta_n\}$ a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} \eta_n < \infty$, and $\{\gamma_n\}$ a sequence of real numbers with $\limsup_{n\longrightarrow\infty} \gamma_n \leq 0$. Suppose that*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n \gamma_n + \eta_n \text{ for all } n \in \mathbb{N}.$$

Then $\lim_{n\longrightarrow\infty} s_n = 0$.

3. MAIN RESULTS

We first consider a convergence result.

Theorem 3.1. *Let C be a closed convex subset of a complete CAT(0) space X, $t : C \longrightarrow C$ be a nonexpansive mapping such that $Fix(t) \neq \emptyset$ and M a positive real number. Suppose $\{\varepsilon_n\}$ and $\{\alpha_n\}$ are sequences in $(0, 1)$ satisfying $\sum_{n=1}^{\infty} \varepsilon_n < \infty$, (C1), (C2) and (C3) respectively. Let $u, z_1 \in C$ be arbitrarily chosen and $\{z_n\}$ be defined by*

$$z_{n+1} = \alpha_n u \oplus (1 - \alpha_n)u_n, \quad u_n \in C$$

such that

$$d(u_n, tz_n) \leq \varepsilon_n M \tag{3.1}$$

for all $n \in \mathbb{N}$. If $\{z_n\}$ is bounded, then the sequence $\{z_n\}$ converges to the unique point of $Fix(t)$ which is nearest to u.

Proof. We follow the proof of Theorem 2.4. By (3.1), we see that

$$\begin{aligned} d(u_n, u_{n+1}) &\leq d(u_n, tz_n) + d(tz_n, tz_{n+1}) + d(tz_{n+1}, u_{n+1}) \\ &\leq d(z_n, z_{n+1}) + M(\varepsilon_n + \varepsilon_{n+1}). \end{aligned}$$

From the definition of z_n, we have

$$\begin{aligned} d(z_{n+1}, z_n) &= d(\alpha_n u \oplus (1-\alpha_n)u_n, \alpha_{n-1} u \oplus (1-\alpha_{n-1})u_{n-1}) \\ &\leq d(\alpha_n u \oplus (1-\alpha_n)u_n, \alpha_n u \oplus (1-\alpha_n)u_{n-1}) \\ &\quad + d(\alpha_n u \oplus (1-\alpha_n)u_{n-1}, \alpha_{n-1} u \oplus (1-\alpha_{n-1})u_{n-1}) \\ &\leq (1-\alpha_n)d(u_n, u_{n-1}) + |\alpha_n - \alpha_{n-1}| d(u, u_{n-1}) \\ &\leq (1-\alpha_n)d(z_n, z_{n-1}) + |\alpha_n - \alpha_{n-1}| d(u, u_{n-1}) \\ &\quad + (1-\alpha_n)M(\varepsilon_n + \varepsilon_{n-1}). \end{aligned}$$

Putting in Lemma 2.6, $[s_n = d(z_n, z_{n-1}), \gamma_n = 0$ and $\eta_n = |\alpha_n - \alpha_{n-1}| d(u, u_{n-1}) + (1-\alpha_n)M(\varepsilon_n + \varepsilon_{n-1})]$ or $[s_n = d(z_n, z_{n-1}), \gamma_n = \left|1 - \frac{\alpha_{n-1}}{\alpha_n}\right| d(u, u_{n-1})$ and $\eta_n = (1-\alpha_n)M(\varepsilon_n + \varepsilon_{n-1})]$ according to $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$ or $\lim_{n \longrightarrow \infty} (\alpha_n / \alpha_{n+1}) = 1$, respectively. Thus, using (C3) and $\sum_{n=1}^{\infty} \varepsilon_n < \infty$, we obtain

$$\lim_{n \longrightarrow \infty} d(z_{n+1}, z_n) = 0.$$

It follows from (C1) that

$$\begin{aligned} d(z_n, u_n) &\leq d(z_n, z_{n+1}) + d(z_{n+1}, u_n) \\ &= d(z_n, z_{n+1}) + d(\alpha_n u \oplus (1-\alpha_n)u_n, u_n) \\ &\leq d(z_n, z_{n+1}) + \alpha_n d(u, u_n) \longrightarrow 0. \end{aligned}$$

This implies

$$\begin{aligned} d(u_n, tu_n) &\leq d(u_n, tz_n) + d(tz_n, tu_n) \\ &\leq \varepsilon_n M + d(z_n, u_n) \longrightarrow 0. \end{aligned}$$

Let $x_s \in [u, tx_s]$ satisfying $d(u, x_s) = s d(u, tx_s)$ for all $s \in (0,1)$. By Theorem 2.3, we have $z =: \lim_{s \longrightarrow 1} x_s$ which is the unique point of $Fix(t)$ nearest to u and $\mu_n(d^2(u,z) - d^2(u, u_n)) \leq 0$ for all Banach limits μ. Moreover, since $d(u_n, u_{n+1}) \leq d(z_n, z_{n+1}) + M(\varepsilon_n + \varepsilon_{n+1}) \longrightarrow 0$,

$$\limsup_{n \longrightarrow \infty} \left(d^2(u,z) - d^2(u,u_n)\right) - \left(d^2(u,z) - d^2(u, u_{n+1})\right) = 0.$$

Therefore Lemma 2.5 implies

$$\limsup_{n \to \infty} \left(d^2(u,z) - (1-\alpha_n)d^2(u,u_n)\right) = \limsup_{n \to \infty} \left(d^2(u,z) - d^2(u,u_n)\right) \leq 0.$$

Consider the following estimates:

$$\begin{aligned} d^2(z_{n+1}, z) &= d^2(\alpha_n u \oplus (1-\alpha_n)u_n, z) \\ &\leq \alpha_n d^2(u,z) + (1-\alpha_n)d^2(u_n, z) - \alpha_n(1-\alpha_n)d^2(u, u_n) \\ &= (1-\alpha_n)d^2(u_n, z) + \alpha_n \left(d^2(u,z) - (1-\alpha_n)d^2(u,u_n)\right) \\ &\leq (1-\alpha_n)(d(u_n, tz_n) + d(tz_n, z))^2 + \alpha_n \left(d^2(u,z) - (1-\alpha_n)d^2(u,u_n)\right) \\ &\leq (1-\alpha_n)(d^2(z_n, z) + 2\varepsilon_n M d(z_n, z) + \varepsilon_n^2 M^2) \\ &\quad + \alpha_n \left(d^2(u,z) - (1-\alpha_n)d^2(u,u_n)\right) \end{aligned}$$

$$\begin{aligned} &= (1-\alpha_n)d^2(z_n,z)+\alpha_n\left(d^2(u,z)-(1-\alpha_n)d^2(u,u_n)\right) \\ &\quad +(1-\alpha_n)(2\varepsilon_n M d(z_n,z)+\varepsilon_n^2M^2) \\ &\le (1-\alpha_n)d^2(z_n,z)+\alpha_n\left(d^2(u,z)-(1-\alpha_n)d^2(u,u_n)\right) \\ &\quad +(1-\alpha_n)(2\varepsilon_n MN+\varepsilon_n^2M^2), \end{aligned}$$

where $N=\sup\{d(z_n,z):n\in\mathbb{N}\}$. We can now use Lemma 2.6 to conclude the proof. □

Here is our first main result.

Theorem 3.2. *Let X be a complete CAT(0) space and $\{A_i : i\in\mathbb{N}\}$ be a family of closed convex subsets of X such that $\bigcap_{i=1}^{\infty} A_i \neq \emptyset$. Let $\{\lambda_n\}$ be a sequence in $(0,1)$ such that $\sum_{n=1}^{\infty}\lambda_n = 1$, $\sum_{i=n}^{\infty}\lambda_i' \longrightarrow 0$ as $n\longrightarrow\infty$ where $\lambda_i' = \sum_{j=i+1}^{\infty}\lambda_j$. Suppose $\{\varepsilon_n\}$ and $\{\alpha_n\}$ are sequences in $(0,1)$ satisfying $\sum_{n=1}^{\infty}\varepsilon_n<\infty$, (C1), (C2) and (C3) respectively. Let $u, z_1\in X$ be arbitrarily chosen and set*

$$r_n=\sup_{i\in\mathbb{N}}\{dist(z_n,A_i)\},\quad \beta_n\in\left(0,\frac{1}{2}\sqrt{4r_n^2+4\varepsilon_n^2}-r_n\right),$$

$$z_{n+1}=\alpha_n u\oplus(1-\alpha_n)u_n,\ \text{where}$$

$$u_n=\bigoplus_{i=1}^{\infty}\lambda_i u_n^{A_i},\quad u_n^{A_i}\in A_i\cap B(z_n:dist(z_n,A_i)+\beta_n^2)$$

for all $n\in\mathbb{N}$. Then the sequence $\{z_n\}$ converges to the unique point of $\bigcap_{i=1}^{\infty}A_i$ which is nearest to u.

Proof. For each $i\in\mathbb{N}$, let $p_i : X\longrightarrow A_i$ be the projection mapping. Using the law of cosine and the definition of β_n, we have

$$\begin{aligned} d^2(u_n^{A_i},p_iz_n) &\le d^2(z_n,u_n^{A_i})-d^2(z_n,p_iz_n) \\ &\le (d(z_n,p_iz_n)+\beta_n)^2-d^2(z_n,p_iz_n) \\ &= 2\beta_n d(z_n,p_iz_n)+\beta_n^2\le\beta_n(2r_n+\beta_n) \\ &< \left(\frac{1}{2}\sqrt{4r_n^2+4\varepsilon_n^2}-r_n\right)\left(\frac{1}{2}\sqrt{4r_n^2+4\varepsilon_n^2}+r_n\right)=\varepsilon_n^2. \end{aligned}$$

Hence $d(u_n^{A_i},p_iz_n)<\varepsilon_n$ for all $n\in\mathbb{N}$. Let $p:X\longrightarrow X$ be defined by

$$px=\bigoplus_{i=1}^{\infty}\lambda_ip_ix$$

for each $x\in X$. From Lemma 2.2, p is nonexpansive and $Fix(p)=\bigcap_{i=1}^{\infty}Fix(p_i)=\bigcap_{i=1}^{\infty}A_i$. For each n, we can choose $m_n\in\mathbb{N}$ such that

$$d\left(\bigoplus_{i=1}^{\infty}\lambda_iu_n^{A_i},\bigoplus_{i=1}^{m_n}\frac{\lambda_i}{\sum_{j=1}^{m_n}\lambda_j}u_n^{A_i}\right)+d\left(\bigoplus_{i=1}^{\infty}\lambda_ip_iz_n,\bigoplus_{i=1}^{m_n}\frac{\lambda_i}{\sum_{j=1}^{m_n}\lambda_j}p_iz_n\right)<\varepsilon_n.$$

Thus

$$\begin{aligned} d(u_n,pz_n) &\le d\left(\bigoplus_{i=1}^{\infty}\lambda_iu_n^{A_i},\bigoplus_{i=1}^{m_n}\frac{\lambda_i}{\sum_{j=1}^{m_n}\lambda_j}u_n^{A_i}\right)+d\left(\bigoplus_{i=1}^{m_n}\frac{\lambda_i}{\sum_{j=1}^{m_n}\lambda_j}u_n^{A_i},\bigoplus_{i=1}^{m_n}\frac{\lambda_i}{\sum_{j=1}^{m_n}\lambda_j}p_iz_n\right) \\ &\quad +d\left(\bigoplus_{i=1}^{m_n}\frac{\lambda_i}{\sum_{j=1}^{m_n}\lambda_j}p_iz_n,\bigoplus_{i=1}^{\infty}\lambda_ip_iz_n\right) \\ &< \sum_{i=1}^{m_n}\frac{\lambda_i}{\sum_{j=1}^{m_n}\lambda_j}d(u_n^{A_i},p_iz_n)+\varepsilon_n<2\varepsilon_n. \end{aligned}$$

Let $q \in \bigcap_{i=1}^{\infty} A_i$. Then

$$\begin{aligned}
d(z_{n+1}, q) &= d(\alpha_n u \oplus (1-\alpha_n)u_n, q) \\
&\leq \alpha_n d(u,q) + (1-\alpha_n) d\left(\bigoplus_{i=1}^{\infty} \lambda_i u_n^{A_i}, q\right) \\
&\leq \alpha_n d(u,q) + (1-\alpha_n) d\left(\bigoplus_{i=1}^{\infty} \lambda_i u_n^{A_i}, \bigoplus_{i=1}^{m_n} \frac{\lambda_i}{\sum_{j=1}^{m_n}\lambda_j} u_n^{A_i}\right) \\
&\quad + (1-\alpha_n) d\left(\bigoplus_{i=1}^{m_n} \frac{\lambda_i}{\sum_{j=1}^{m_n}\lambda_j} u_n^{A_i}, q\right) \\
&\leq \alpha_n d(u,q) + (1-\alpha_n)\left(\varepsilon_n + \sum_{i=1}^{m_n} \frac{\lambda_i}{\sum_{j=1}^{m_n}\lambda_j}(d(u_n^{A_i}, p_i z_n) + d(p_i z_n, q))\right) \\
&\leq \alpha_n d(u,q) + (1-\alpha_n) d(z_n, q) + 2(1-\alpha_n)\varepsilon_n \\
&\leq \max\{d(u,q), d(z_n,q)\} + 2(1-\alpha_n)\varepsilon_n.
\end{aligned}$$

By induction we have

$$d(z_{n+1}, q) \leq \max\{d(u,q), d(z_1,q)\} + 2\sum_{n=1}^{\infty}(1-\alpha_n)\varepsilon_n < \infty \text{ for all } n \in \mathbb{N}.$$

This implies the sequence $\{z_n\}$ is bounded. The result now follows from Theorem 3.1. □

When the domain is bounded, we have the following result where the sequence $\{z_n\}$ is computable.

Theorem 3.3. *Let X be a complete CAT(0) space and $\{A_i : i \in \mathbb{N}\}$ be a family of closed convex subsets of X such that $\bigcap_{i=1}^{\infty} A_i \neq \emptyset$ and $\bigcup_{i=1}^{\infty} A_i$ is bounded. Let $\{\lambda_n\}$ be a sequence in $(0,1)$ such that $\sum_{n=1}^{\infty}\lambda_n = 1$, $\sum_{i=n}^{\infty}\lambda_i' \longrightarrow 0$ as $n \longrightarrow \infty$ where $\lambda_i' = \sum_{j=i+1}^{\infty}\lambda_j$. Let $\{\varepsilon_n\}$ be a sequence in $(0, \frac{1}{2})$ and $\{\alpha_n\}$ be a sequence in $(0,1)$ satisfying $\sum_{n=1}^{\infty}\varepsilon_n < \infty$, (C1), (C2) and (C3) respectively. Let $u, z_1 \in C$ be arbitrarily chosen. For each $n \in \mathbb{N}$, choose $k_n \in \mathbb{N}$ such that $\lambda_i' < \varepsilon_n$ for all $i \geq k_n$ and set*

$$r_n = \sup_{i\in\mathbb{N}}\{dist(z_n, A_i)\}, \quad \beta_n \in \left(0, \frac{1}{2}\sqrt{4r_n^2 + 4\varepsilon_n^2} - r_n\right),$$

$$z_{n+1} = \alpha_n u \oplus (1-\alpha_n)u_n', \text{ where}$$

$$u_n' = \bigoplus_{i=1}^{k_n} \frac{\lambda_i}{\sum_{j=1}^{k_n}\lambda_j} u_n^{A_i}, \quad u_n^{A_i} \in A_i \cap B(z_n : dist(z_n, A_i) + \beta_n^2).$$

Then the sequence $\{z_n\}$ converges to the unique point of $\bigcap_{i=1}^{\infty} A_i$ which is nearest to u.

Proof. Let p_i and p be as in the proof of Theorem 3.2. Thus we have

$$d(u_n^{A_i}, p_i z_n) < \varepsilon_n$$

for all $n \in \mathbb{N}$. For each n, we can choose $m_n > k_n$ such that

$$d\left(\bigoplus_{i=1}^{\infty}\lambda_i p_i z_n, \bigoplus_{i=1}^{m_n}\frac{\lambda_i}{\sum_{j=1}^{m_n}\lambda_j} p_i z_n\right) < \varepsilon_n.$$

Since $\lambda_i' < \varepsilon_n < \frac{1}{2}$, we have

$$d\left(\bigoplus_{i=1}^{k_n}\frac{\lambda_i}{\sum_{j=1}^{k_n}\lambda_j} p_i z_n, \bigoplus_{i=1}^{m_n}\frac{\lambda_i}{\sum_{j=1}^{m_n}\lambda_j} p_i z_n\right)$$

$$\leq d\left(\bigoplus_{i=1}^{k_n}\frac{\lambda_i}{\sum_{j=1}^{k_n}\lambda_j}p_iz_n,\bigoplus_{i=1}^{k_n+1}\frac{\lambda_i}{\sum_{j=1}^{k_n+1}\lambda_j}p_iz_n\right)+\cdots+d\left(\bigoplus_{i=1}^{m_n-1}\frac{\lambda_i}{\sum_{j=1}^{m_n-1}\lambda_j}p_iz_n,\bigoplus_{i=1}^{m_n}\frac{\lambda_i}{\sum_{j=1}^{m_n}\lambda_j}p_iz_n\right)$$

$$\leq \frac{\lambda_{k_n+1}}{\sum_{j=1}^{k_n+1}\lambda_j}d\left(\bigoplus_{i=1}^{k_n}\frac{\lambda_i}{\sum_{j=1}^{k_n}\lambda_j}p_iz_n,p_{k_n+1}z_n\right)+\cdots+\frac{\lambda_{m_n}}{\sum_{j=1}^{m_n}\lambda_j}d\left(\bigoplus_{i=1}^{m_n-1}\frac{\lambda_i}{\sum_{j=1}^{m_n-1}\lambda_j}p_iz_n,p_{m_n}z_n\right)$$

$$\leq K\sum_{i=k_n+1}^{m_n}\frac{\lambda_i}{1-\lambda_i'}<2K\sum_{i=k_n+1}^{m_n}\lambda_i<2K\lambda'_{k_n+1}<2K\varepsilon_n,$$

where $K=\sup_{n\in\mathbb{N}}\left\{\sup_{l\in\mathbb{N}}\left\{d\left(\bigoplus_{i=1}^{l}\frac{\lambda_i}{\sum_{j=1}^{l}\lambda_j}p_iz_n,p_{l+1}z_n\right)\right\}\right\}<\infty.$
Thus

$$d(u_n',pz_n)\leq\varepsilon_n(2K+2).$$

The result now follows from Theorem 3.1. □

As corollaries, with the same lines of proofs, the corresponding results hold for a finite family $\{t_i : i=1,2,...,N\}$ of mappings.

Applications

Let X be a complete CAT(0) space. For a function $h : X \longrightarrow (-\infty,\infty]$, the $\alpha-$sublevel set is defined by

$$A_h^\alpha=\{x\in X : h(x)\leq\alpha\}.$$

Let $\{h_i : i\in\mathbb{N}\}$ be a family of lower semi-continuous and convex functions from X into $(-\infty,\infty]$. Bačák, Searston and Sims [2] introduced the method for approximating a minimizer of the functional $H : X \longrightarrow (-\infty,\infty]$, where $H=\sup_{i\in\mathbb{N}}h_i$ as the following:

Proposition 3.4. [2, Proposition 5.2] *Let X be a complete CAT(0) space and a mapping $F : X\longrightarrow(-\infty,\infty]$ be of the form $F=\max\{f,g\}$, where $f,g : X\longrightarrow(-\infty,\infty]$ are lower semi-continuous and convex functions. Let $\alpha>\inf_{x\in X}F(x)>-\infty$, and A_F^α be nonempty. Assume that f is both uniformly convex and uniformly continuous on bounded sets of X. Let $x_0\in X$ be a starting point and $\{x_n\}\subset X$ be the sequence generated by*

$$x_{2n-1}=P_f(x_{2n-1}),\quad x_{2n}=P_g(x_{2n-1}),\ n\in\mathbb{N},$$

where P_f and P_g are projection mappings from X to A_f^α and A_g^α respectively. Then $\{x_n\}$ converges to $z\in A_F^\alpha$.

We now show Propositions providing the strong convergence of the sequence $\{z_n\}$ to an (approximative) minimizer of the functional H.

Proposition 3.5. *Let X be a complete CAT(0) space and a mapping $H : X\longrightarrow(-\infty,\infty]$ be of the form $H=\sup_{i\in\mathbb{N}}h_i$, where $h_i : X\longrightarrow(-\infty,\infty]$ are lower semi-continuous and convex functions for all $i\in\mathbb{N}$. Let $\alpha>\inf_{x\in X}H(x)>-\infty$. Let $\{\lambda_n\}$ be a sequence in $(0,1)$ such that $\sum_{n=1}^\infty\lambda_n=1$, $\sum_{i=n}^\infty\lambda_i'\longrightarrow0$ as $n\longrightarrow\infty$ where $\lambda_i'=\sum_{j=i+1}^\infty\lambda_j$. Let $\{\varepsilon_n\}$ and $\{\alpha_n\}$ be sequences in $(0,1)$ satisfying $\sum_{n=1}^\infty\varepsilon_n<\infty$, (C1), (C2) and (C3) respectively. Let $u,z_1\in X$ are arbitrarily chosen and set*

$$r_n=\sup_{i\in\mathbb{N}}\{dist(z_n,A_{h_i}^\alpha)\},\quad\beta_n\in\left(0,\frac{1}{2}\sqrt{4r_n^2+4\varepsilon_n^2}-r_n\right),$$

$$z_{n+1}=\alpha_nu\oplus(1-\alpha_n)u_n,$$

where

$$u_n = \bigoplus_{i=1}^{\infty} \lambda_i u_n^i, \ \ u_n^i \in A_{h_i}^\alpha \cap B(z_n : dist(z_n, A_{f_i}^\alpha) + \beta_n^2)$$

for all $n \in \mathbb{N}$. *Then the sequence* $\{z_n\}$ *converges to the unique point of* A_H^α *which is nearest to* u.

Proof. Since $h_i : X \longrightarrow (-\infty, \infty]$ are lower semi-continuous and convex functions, $A_{h_i}^\alpha$ is closed and convex for all $i \in \mathbb{N}$. The result then follows from Theorem 3.2. □

Proposition 3.6. *Let* X *be a complete CAT(0) space and a mapping* $H : X \longrightarrow (-\infty, \infty]$ *be of the form* $H = \sup_{i\in\mathbb{N}} h_i$, *where* $h_i : X \longrightarrow (-\infty, \infty]$ *are lower semi-continuous and convex functions for all* $i \in \mathbb{N}$. *Let* $\alpha > \inf_{x\in X} H(x) > -\infty$. *Let* $\{\lambda_n\}$ *be a sequence in* $(0, 1)$ *such that* $\sum_{n=1}^{\infty} \lambda_n = 1$, $\sum_{i=n}^{\infty} \lambda_i' \longrightarrow 0$ *as* $n \longrightarrow \infty$ *where* $\lambda_i' = \sum_{j=i+1}^{\infty} \lambda_j$. *Let* $\{\varepsilon_n\}$ *be a sequence in* $(0, \frac{1}{2})$ *and* $\{\alpha_n\}$ *be a sequence in* $(0, 1)$ *satisfying* $\sum_{n=1}^{\infty} \varepsilon_n < \infty$, *(C1), (C2) and (C3) respectively. Let* $u, z_1 \in C$ *be arbitrarily chosen. For each* $n \in \mathbb{N}$, *choose* $k_n \in \mathbb{N}$ *such that* $\lambda_i' < \varepsilon_n$ *for all* $i \geq k_n$ *and set*

$$r_n = \sup_{i\in\mathbb{N}} \{dist(z_n, A_{h_i}^\alpha)\}, \quad \beta_n \in \left(0, \frac{1}{2}\sqrt{4r_n^2 + 4\varepsilon_n^2} - r_n\right),$$

$$z_{n+1} = \alpha_n u \oplus (1 - \alpha_n) u_n',$$

where

$$u_n' = \bigoplus_{i=1}^{k_n} \frac{\lambda_i}{\sum_{j=1}^{k_n} \lambda_j} u_n^i, \ \ u_n^i \in A_{h_i}^\alpha \cap B(z_n : dist(z_n, A_{h_i}^\alpha) + \beta_n^2).$$

If $\{z_n\}$ *is bounded, then the sequence* $\{z_n\}$ *converges to the unique point of* A_H^α *which is nearest to* u.

Proof. Here we apply Theorem 3.3. □

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