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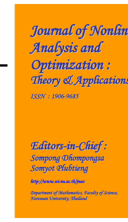

A GENESIS OF GENERAL KKM THEOREMS FOR ABSTRACT CONVEX SPACES

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ABSTRACT. In a recent work of the author [42], we obtained three general KKM type theorems for abstract convex spaces. In this review, we show that two of them can be stated for intersectionally closed-valued KKM maps in the sense of Luc et al. [13]. Each of such KKM type theorems contains a large number of previously known particular forms. We recall some of them due to the author in order to give a short history on such theorems. Further comments on related works are given.

KEYWORDS : Abstract convex space; (partial) KKM principle; KKM space; Generalized (G-) convex space; Fixed point.

1. INTRODUCTION

Many problems in nonlinear analysis can be solved by showing the non-emptiness of the intersection of certain family of subsets of an underlying set. One of the remarkable results on such nonempty intersection is the celebrated Knaster-Kuratowski-Mazurkiewicz theorem (simply, the KKM theorem) in 1929 [10], which is concerned with certain types of multimaps later called the KKM maps. The KKM theory, first named this by the author [17], is the study of applications of equivalent formulations or generalizations of the KKM theorem.

Since 2006, we have introduced the new concepts of abstract convex spaces and KKM spaces which are adequate to establish the KKM theory. With such new concepts, we could generalize and simplify many known results in the theory on convex spaces, H-spaces, G-convex spaces, and others; see [27-33,39,40,42,43].

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In a recent work [42], we reviewed some known facts on abstract convex spaces and obtained three general KKM type theorems which are equivalent or can be extended to Theorems A, B, and C in this paper, resp. Each of them contains a large number of previously known particular forms which are generalizations, imitations, or modifications of the original KKM theorem due to many other authors. In the present paper, we recall some historically important previous particular versions of our KKM type theorems in [42] in order to give a short history on each of them. Moreover, further comments on related works are given.

2. ABSTRACT CONVEX SPACES

For the concepts of abstract convex spaces and KKM spaces, the reader may consult with our previous works [27-33,39,40,42,43].

Definition 2.1. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D\rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$, where $\langle D\rangle$ is the set of all nonempty finite subsets of D.

For any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_A \mid A \in \langle D'\rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D'\rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

When $D \subset E$, a subset X of E is said to be Γ-*convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Definition 2.2. Let $(E, D; \Gamma)$ be an abstract convex space and Z a topological space. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all } A \in \langle D\rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

A multimap $F : E \multimap Z$ is called a $\mathfrak{KC}$-*map* [resp., a $\mathfrak{KO}$-*map*] if, for any closed-valued [resp., open-valued] KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. In this case, we denote $F \in \mathfrak{KC}(E, D, Z)$ [resp., $F \in \mathfrak{KO}(E, D, Z)$]. Some authors use KKM instead of $\mathfrak{KC}$.

We have abstract convex subspaces as the following simple observation shows:

Proposition 2.3. ([42]) *For an abstract convex space $(E, D; \Gamma)$ and a nonempty subset D' of D, let X be a Γ-convex subset of E relative to D' and $\Gamma' : \langle D'\rangle \multimap X$ a map defined by*

$$\Gamma'_A := \Gamma_A \subset X \ for\ A \in \langle D'\rangle.$$

Then $(X, D'; \Gamma')$ itself is an abstract convex space called a subspace relative to D'.

Proposition 2.4. ([42]) *Let $(E, D; \Gamma)$ be an abstract convex space, $(X, D'; \Gamma')$ a subspace, and Z a topological space. If $F \in \mathfrak{KC}(E, D, Z)$, then $F|_X \in \mathfrak{KC}(X, D', \overline{F(X)})$.*

Definition 2.5. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement $1_E \in \mathfrak{KC}(E, D, E)$; that is, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. The *KKM principle* is the statement $1_E \in \mathfrak{KC}(E, D, E) \cap \mathfrak{KO}(E, D, E)$; that is, the same property also holds for any open-valued KKM map.

An abstract convex space is called a *KKM space* if it satisfies the KKM principle.

We had the following diagram for triples $(E, D; \Gamma)$:

Simplex $\Longrightarrow$ Convex subset of a t.v.s. $\Longrightarrow$ Lassonde type convex space
$\Longrightarrow$ H-space $\Longrightarrow$ G-convex space $\Longleftrightarrow$ ϕ_A-space $\Longrightarrow$ KKM space
$\Longrightarrow$ Space satisfying the partial KKM principle
$\Longrightarrow$ Abstract convex space.

Example. There are plenty of examples of abstract convex spaces; see [31-33,39,40, 42,43]. Here we need only two classes of them:

(I) A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ due to Park is an abstract convex space such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \longrightarrow \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, Δ_n is the standard n-simplex with vertices $\{e_i\}_{i=0}^n$ and Δ_J its face corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$, then $\Delta_J = \operatorname{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$.

(II) A *ϕ_A-space* $(X, D; \{\phi_A\}_{A \in \langle D \rangle})$ consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \longrightarrow X$ (that is, singular n-simplexes) for $A \in \langle D \rangle$ with $|A| = n + 1$. Every ϕ_A-space can be made into a G-convex space; see [28,41].

3. THE KKM THEOREM A

The following whole intersection property for the map-values of a KKM map is a standard form of the KKM type theorems:

Theorem A. *Let $(E, D; \Gamma)$ be an abstract convex space, the identity map $1_E \in \mathfrak{KC}(E, D, E)$ [resp., $1_E \in \mathfrak{KO}(E, D, E)$], and $G : D \multimap E$ a multimap satisfying*

(1) *G has closed [resp., open] values; and*

(2) *$\Gamma_N \subset G(N)$ for any $N \in \langle D \rangle$ (that is, G is a KKM map).*

Then $\{G(z)\}_{z \in D}$ has the finite intersection property.

Further, if

(3) *$\bigcap_{z \in M} \overline{G(z)}$ is compact for some $M \in \langle D \rangle$,*

then we have

$$\bigcap_{y \in D} \overline{G(y)} \neq \emptyset.$$

Proof. The first part is a simple consequence of definition. For the second part, let $K := \bigcap_{z \in M} \overline{G(z)}$. Since $\{G(z) \mid z \in D\}$ has the finite intersection property, so does $\{K \cap \overline{G(z)} \mid z \in D\}$ in the compact set K. Hence it has the whole intersection property. □

Recall that Theorem A is a simple consequence of the definitions of the partial KKM principle or the KKM space.

From now on, in this section, we give historically well-known particular forms of Theorem A in the chronological order.

(I) Knaster, Kuratowski, and Mazurkiewicz in 1929 [10] obtained the following so-called KKM theorem from the Sperner combinatorial lemma, and applied it to a simple proof of the Brouwer fixed point theorem:

Theorem. ([10]) *Let A_i $(0 \leq i \leq n)$ be $n+1$ closed subsets of an n-simplex $p_0 p_1 \cdots p_n$. If the inclusion relation*

$$p_{i_0} p_{i_1} \cdots p_{i_k} \subset A_{i_0} \cup A_{i_1} \cup \cdots \cup A_{i_k}$$

holds for all faces $p_{i_0} p_{i_1} \cdots p_{i_k}$ $(0 \leq k \leq n,\ 0 \leq i_0 < i_1 < \cdots < i_k \leq n)$, then $\bigcap_{i=0}^{n} A_i \neq \emptyset$.

(II) A milestone on the history of the KKM theory was erected by Ky Fan in 1961 [3]. He extended the KKM theorem to arbitrary topological vector spaces and applied it to coincidence theorems generalizing the Tychonoff fixed point theorem and a result concerning two continuous maps from a compact convex set into a uniform space.

Lemma. ([3]) *Let X be an arbitrary set in a topological vector space Y. To each $x \in X$, let a closed set $F(x)$ in Y be given such that the following two conditions are satisfied:*

(i) *The convex hull of any finite subset $\{x_1, x_2, \cdots, x_n\}$ of X is contained in $\bigcup_{i=1}^{n} F(x_i)$.*

(ii) *$F(x)$ is compact for at least one $x \in X$.*

Then $\bigcap_{x \in X} F(x) \neq \emptyset$.

This is usually known as the KKMF theorem and was shown to have numerous applications; see [24].

(III) In 1987, W.K. Kim [8] and M.-H. Shih and K.-K. Tan [51] independently showed that any simplex is a KKM space:

Theorem. ([8,51]) *Let $X = \{x_1, \ldots, x_n\}$ be the set of vertices of a simplex S^{n-1} in $E = \mathbb{R}^n$ and let $F : X \multimap E$ be an open-valued KKM map. Then $\bigcap_{i=1}^{n} F(x_i) \neq \emptyset$.*

However, the main idea was given in the earlier work of Fan [4, Theorem 2] as a matching theorem for closed coverings. Later, results of Kim [8,9] were generalized by the present author [15,16]. Moreover, Lassonde [12] refined Kim's idea and gave some applications.

(IV) The following shows that G-convex spaces are KKM spaces:

Theorem. ([25,26]) *Let $(X, D; \Gamma)$ be a G-convex space and $F : D \multimap X$ a map such that*

(1) *F has closed [resp., open] values; and*

(2) *F is a KKM map.*

Then $\{F(z)\}_{z \in D}$ has the finite intersection property.

Further, if

(3) *$\bigcap_{z \in M} \overline{F(z)}$ is compact for some $M \in \langle D \rangle$,*

then we have

$$\bigcap_{z \in D} \overline{F(z)} \neq \emptyset.$$

(V) In 2008, we showed that any ϕ_A-space is a KKM space [34-36]:

Definition. For a ϕ_A-space $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ and a topological vector space Z, let $F : X \multimap Z$ be a multimap. Then a map $G : D \multimap X$ satisfying

$$F(\phi_A(\Delta_J)) \subset G(J) \text{ for each } A \in \langle D\rangle \text{ and } J \in \langle A\rangle$$

is called a *KKM map w.r.t. F*.

Theorem. ([34-36]) *For a ϕ_A-space $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$, let $G : D \multimap X$ be a KKM map with closed [resp., open] values. Then $\{G(z)\}_{z\in D}$ has the finite intersection property. (More precisely, for each $N \in \langle D\rangle$ with $|N| = n+1$, we have $\phi_N(\Delta_n) \cap \bigcap_{z\in N} G(z) \neq \emptyset$.)*

Further, if

(3) $\bigcap_{z\in M} \overline{G(z)}$ *is compact for some $M \in \langle D\rangle$,*

then we have $\bigcap_{z\in D} \overline{G(z)} \neq \emptyset$.

4. THE KKM THEOREM B

Recall that the main conclusions of most results in the preceding section are of the form

$$\bigcap_{y\in D} \overline{G(y)} \neq \emptyset.$$

for a multimap $G : D \multimap E$.

Consider the following related four conditions:

(a) $\bigcap_{z\in D} \overline{G(z)} \neq \emptyset$ implies $\bigcap_{z\in D} G(z) \neq \emptyset$.

(b) $\bigcap_{z\in D} \overline{G(z)} = \overline{\bigcap_{z\in D} G(z)}$ (G is *intersectionally closed-valued* [13]).

(c) $\bigcap_{z\in D} \overline{G(z)} = \bigcap_{z\in D} G(z)$ (G is *transfer closed-valued*).

(d) G is closed-valued.

In [13], the authors noted that (a) $\Longleftarrow$ (b) $\Longleftarrow$ (c) $\Longleftarrow$ (d), and gave examples of multimaps satisfying (b) but not (c). Therefore it is a proper time to deal with condition (b) instead of (c) in the KKM theory.

Example. The following maps G are intersectionally closed-valued, but not transfer closed-valued:

(1) $G(z)$ = (0, 1) for every $z \in$ [0, 1] is a constant multimap from D = $[0, 1]$ to E = [0, 1]; see [13].

(2) Each $G(z)$ is a convex set in a Euclidean space and has a relative interior point in common; see Rockafellar [50, Theorem 6.5].

(3) For a given subset E of a topological vector space with $x^* \in E$, each $G(z), z \in D$, is a nicely star-shaped at x^*; see [13].

From the partial KKM principle we have a whole intersection property of the Fan type as follows:

Theorem B. *Let $(E, D; \Gamma)$ be an abstract convex space satisfying the partial KKM principle [that is, $1_E \in \mathfrak{KC}(E, D, E)$] and $G : D \multimap E$ a map such that*

(1) *$\overline{G}$ is a KKM map [that is, $\Gamma_A \subset \overline{G}(A)$ for all $A \in \langle D\rangle$]; and*

(2) *there exists a nonempty compact subset K of E such that either*

(i) $\bigcap\{\overline{G(z)} \mid z \in M\} \subset K$ *for some* $M \in \langle D \rangle$*; or*

(ii) *for each* $N \in \langle D \rangle$*, there exists a compact* Γ*-convex subset* L_N *of* E *relative to some* $D' \subset D$ *such that* $N \subset D'$ *and*

$$L_N \cap \bigcap_{z \in D'} \overline{G(z)} \subset K.$$

Then we have $K \cap \bigcap_{z \in D} \overline{G(z)} \neq \emptyset$.

Furthermore,

(α) *if* G *is transfer closed-valued, then* $K \cap \bigcap\{G(z) \mid z \in D\} \neq \emptyset$*;*

(β) *if* G *is intersectionally closed-valued, then* $\bigcap\{G(z) \mid z \in D\} \neq \emptyset$.

Proof. As in [33,42,43], from the hypothesis, we must have $K \cap \bigcap_{z \in D} \overline{G(z)} \neq \emptyset$.

(α) Since G is transfer closed-valued,

$$K \cap \bigcap_{z \in D} G(z) = K \cap \bigcap_{z \in D} \overline{G(z)} \neq \emptyset.$$

(β) Since G is intersectionally closed-valued,

$$\overline{\bigcap_{z \in D} G(z)} = \bigcap_{z \in D} \overline{G(z)} \neq \emptyset.$$

This implies the conclusion. □

Recall that conditions (i) and (ii) in Theorem B are usually called the *compactness conditions* or the *coercivity conditions*, and (ii) has numerous variations or particular forms appeared in a very large number of literature. Note that Theorem B can be easily deduced from the compact case of Theorem A, and hence it seems to be not a big problem to treat the case (ii).

From now on, in this section, we give some important forerunners of Theorem B:

(I) According to Lassonde [11], a *convex space* is a nonempty convex set (in a vector space) equipped with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. Such convex hulls are called *polytopes*. A nonempty subset L of a convex space X is called a *c-compact set* if for each finite subset $S \subset X$ there is a compact convex set $L_S \subset X$ such that $L \cup S \subset L_S$.

In 1983, Lassonde gave the following:

Theorem. ([11]) *Let D be an arbitrary set in a convex space X, Y any topological space, and* $F : D \longrightarrow 2^Y$ *a multifunction having the following properties*

(i) *For each* $x \in D$, $F(x)$ *is compactly closed in Y.*

(ii) *For some continuous map* $s : X \longrightarrow Y$*, the multifunction* $G : D \longrightarrow 2^X$ *given by* $G(x) = s^{-1}(F(x))$ *is KKM.*

(iii) *For some c-compact set* $L \subset X$, $\bigcap\{F(x) \mid x \in L \cap D\}$ *is compact.*

Then $\bigcap\{F(x) \mid x \in D\} \neq \emptyset$.

Note that (iii) is a compactness condition implying condition (ii) of Theorem B.

In our work in 1989 [14], from this, we deduced a general Fan-Browder fixed point theorem with its various applications to analytic alternatives, section properties, fixed points, minimax and variational inequalities.

More general results for H-spaces $(X, D; \Gamma)$ are deduced in 1992 [18].

Note that the continuous maps s were later extended to acyclic maps, admissible maps, better admissible maps, and $\mathfrak{KC}$-maps by the author; see Section 5.

(II) In 1984, Fan obtained the following:

Theorem. ([4]) *In a topological vector space, let Y be a convex set and $\emptyset \neq X \subset Y$. For each $x \in X$, let $F(x)$ be a relatively closed subset of Y such that the convex hull of every finite subset $\{x_1, x_2, \ldots, x_n\}$ of X is contained in the corresponding union $\bigcup_{i=1}^{n} F(x_i)$. If there is a nonempty subset X_0 of X such that the intersection $\bigcap_{x \in X_0} F(x)$ is compact and X_0 is contained in a compact convex subset of Y, then $\bigcap_{x \in X} \neq \emptyset$.*

(III) A particular form of Theorem B for H-spaces $(X, D; \Gamma)$ was obtained and applied in 1993 [19].

(IV) In 2000 [26], for a G-convex space $(X \supset D; \Gamma)$, we had a particular form of Theorem B as follows:

Theorem. ([26]) *Let $(X \supset D; \Gamma)$ be a G-convex space, K a nonempty compact subset of X, and $F : D \multimap X$ a multimap such that*

(1) *F is transfer closed-valued;*

(2) *$\overline{F}$ is a KKM map; and*

(3) *for each $N \in \langle D \rangle$, there exists a compact G-convex subspace L_N of X containing N such that*

$$L_N \cap \bigcap \{\overline{F(z)} \mid z \in L_N \cap D\} \subset K.$$

Then $K \cap \bigcap \{F(z) \mid z \in D\} \neq \emptyset$.

Recall that this theorem generalizes earlier works of Tian, Ding, Chang et al., Lin and Park; see [26]. For H-spaces, the preceding theorem was given in [18,19]. Note that many particular forms are still happening. Moreover, condition (3) was first due to S.-Y. Chang [2] as a generalizations of Fan's original conditions, and had been adopted by the present author since 1992. However, still many authors are using particular forms of (3).

(V) In 2001 [49], Park and Lee defined generalized KKM maps on G-convex spaces. In 2007 [7], H. Kim and the author showed that a generalized KKM map G is a KKM map on a new G-convex space $(X, I; \Gamma^G)$ and, from Theorem B, deduced a KKM type theorem [42, Theorem 3.7] for generalized KKM maps with closed values.

(VI) When G is closed-valued or transfer closed-valued in condition (1) of Theorem B, we obtained the following already:

Theorem. ([42]) *Let $(X, D; \Gamma)$ be an abstract convex space satisfying the partial KKM principle, and $G : D \multimap X$ a map such that*

(1) *G is transfer closed-valued;*

(2) *$\overline{G}$ is a KKM map; and*

(3) *there exists a nonempty compact subset K of X such that either*

(i) *$K \supset \bigcap \{\overline{G(z)} \mid z \in M\}$ for some $M \in \langle D \rangle$; or*

(ii) *for each $N \in \langle D \rangle$, there exists a compact Γ-convex subset L_N of X relative to some $D' \subset D$ such that $N \subset D'$ and*

$$K \supset L_N \cap \bigcap \{\overline{G(z)} \mid z \in D'\}.$$

Then $K \cap \bigcap\{G(z) \mid z \in D\} \neq \emptyset$.

This is an an equivalent form of [33, Theorem 8.2], subsumes a very large number of particular KKM type theorems in the literature, and has a number of equivalent formulations for abstract convex spaces satisfying the partial KKM principle as in [33]. Since the preceding theorem can be easily deduced from Theorem B, we do not need to think about the 'transfer' case.

5. THE KKM THEOREM C

Theorem B can be extended to $F \in \mathfrak{KC}(X, D, Z)$ instead of $1_X \in \mathfrak{KC}(X, D, X)$) as the following generalized form of [42, Theorem 2.10] shows:

Theorem C. *Let* $(X, D; \Gamma)$ *be an abstract convex space,* Z *a topological space,* $F \in \mathfrak{KC}(X, D, Z)$*, and* $G : D \multimap Z$ *a map such that*

(1) $\overline{G}$ *is a KKM map w.r.t.* F*; and*

(2) *there exists a nonempty compact subset* K *of* Z *such that either*

(i) $K \supset \bigcap\{\overline{G(y)} \mid y \in M\}$ *for some* $M \in \langle D\rangle$*; or*

(ii) *for each* $N \in \langle D\rangle$*, there exists a* Γ*-convex subset* L_N *of* X *relative to some* $D' \subset D$ *such that* $N \subset D'$*,* $F(L_N)$ *is compact, and*

$$K \supset \overline{F(L_N)} \cap \bigcap\{\overline{G(z)} \mid z \in D'\}.$$

Then we have

$$\overline{F(X)} \cap K \cap \bigcap_{y\in D} \overline{G(y)} \neq \emptyset.$$

Furthermore,

(α) *if* G *is transfer closed-valued, then* $\overline{F(X)} \cap K \cap \bigcap\{G(z) \mid z \in D\} \neq \emptyset$*; and*

(β) *if* G *is intersectionally closed-valued, then* $\bigcap\{G(z) \mid z \in D\} \neq \emptyset$.

Proof. Case (i): Since $F(\Gamma_N) \subset \overline{G}(N)$ for each $N \in \langle D\rangle$ by (1), we have

$$F(\Gamma_N) \subset F(X) \cap \overline{G}(N) \subset \overline{F(X)} \cap \overline{G}(N) =: G'(N),$$

where $G'(y) := \overline{F(X)} \cap \overline{G(y)}$ is closed for each $y \in D$. Then, by Proposition 2.4 on $(X, D', \overline{F(X)})$, $\{G'(y) \mid y \in D\}$ has the finite intersection property. Since the requirement (i) implies

$$\overline{F(X)} \cap K \supset \overline{F(X)} \cap \bigcap_{y\in M} \overline{G(y)} = \bigcap_{y\in M} G'(y),$$

$\bigcap_{y\in M} G'(y)$ is compact. Therefore $\bigcap\{G'(y) \mid y \in D\} \neq \emptyset$ by Theorem A and hence

$$\overline{F(X)} \cap K \cap \bigcap_{y\in D} \overline{G(y)} \neq \emptyset.$$

Case (ii): Suppose that

$$\overline{F(X)} \cap K \cap \bigcap_{y\in D} \overline{G(y)} = \emptyset.$$

Since $\overline{F(X)} \cap K$ is compact, $\overline{F(X)} \cap K \subset \bigcup\{Z \setminus \overline{G(y)} \mid y \in N\}$ for some $N \in \langle D\rangle$. Let L_N be the Γ-convex subset of X in (ii). Define $G' : D' \multimap \overline{F(L_N)}$ by

$G'(y) := \overline{G(y)} \cap \overline{F(L_N)}$ for $y \in D'$. For each $A \in \langle D' \rangle$, define $\Gamma'_A := \Gamma_A \cap L_N$. Then $(L_N, D'; \Gamma')$ is an abstract convex space. Moreover,

$$(F|_{L_N})(\Gamma'_A) \subset F(\Gamma_A) \cap F(L_N) \subset \overline{G}(A) \cap \overline{F(L_N)} = G'(A)$$

for each $A \in \langle D' \rangle$ by (2); and hence $G' : D' \multimap \overline{F(L_N)}$ is a KKM map w.r.t. $F|_{L_N}$ on the abstract convex space $(L_N, D'; \Gamma')$ with closed values in $\overline{F(L_N)}$. Since $F \in \mathfrak{KC}(X, D, Z)$, by Proposition 2.4, we have $F|_{L_N} \in \mathfrak{KC}(L_N, D', \overline{F(L_N)})$ and hence, $\{G'(y) \mid y \in D'\} = \{\overline{G(y)} \cap \overline{F(L_N)} \mid y \in D'\}$ has the finite intersection property. Since we assumed that $\overline{F(L_N)}$ is compact, each $G'(y)$ is compact. Hence $\bigcap\{G'(y) \mid y \in D'\} \neq \emptyset$ by Theorem A and there exists a

$$z \in \bigcap_{y \in D'} G'(y) = \overline{F(L_N)} \cap \bigcap_{y \in D'} \overline{G(y)} \subset K$$

by (ii). Since $z \in K$ and $z \in \overline{F(L_N)}$, we have $z \in \bigcup\{Z \setminus \overline{G(y)} \mid y \in N\}$ by our assumption. So $z \notin \overline{G(y)}$ for some $y \in N \subset D'$, and hence $z \notin \bigcap\{\overline{G(y)} \mid y \in D'\}$. This contradicts $z \in \bigcap\{G'(y) \mid y \in D'\}$. Therefore, we must have

$$\overline{F(X)} \cap K \cap \bigcap_{y \in D} \overline{G(y)} \neq \emptyset.$$

(α) Since G is transfer closed-valued,

$$\overline{F(X)} \cap K \cap \bigcap_{y \in D} G(y) = \overline{F(X)} \cap K \cap \bigcap_{y \in D} \overline{G(y)} \neq \emptyset.$$

(β) Since G is intersectionally closed-valued,

$$\overline{\bigcap_{y \in D} G(y)} = \bigcap_{y \in D} \overline{G(y)} \neq \emptyset.$$

This implies the conclusion. □

Note that Theorem C can be reformulated to the equivalent forms of coincidence theorems, matching theorems, analytic alternatives, minimax inequalities, geometric and section properties as in [7,33,48].

Let $cc(E)$ denote the set of nonempty closed convex subsets of a convex space E, $kc(E)$ the set of nonempty compact convex subsets of E, and $ka(E)$ the set of compact acyclic subsets of a topological space E.

Now we give a brief genesis of Theorem C as follows:

(I) In 1992, for a convex space and an acyclic map (that is, a u.s.c. map with compact acyclic values), we have the following KKM theorem [17, Theorem 3]:

Theorem. ([17]) *Let D be a nonempty subset of a convex space X, Y a Hausdorff space, $F : X \longrightarrow ka(Y)$ a u.s.c. multimap, and K a nonempty compact subset of Y. Let $G : D \multimap Y$ be a multimap such that*

(1) *for each $x \in D$, $G(x)$ is closed;*

(2) *for each $N \in \langle D \rangle$, $F(\mathrm{co}\, N) \subset G(N)$; and*

(3) *there exists an $L_N \in kc(X)$ containing N such that $x \in L_N \setminus F^+(K)$ implies $\bigcap\{G(z) \mid z \in L_N \cap D\} \subset Y \setminus F(x)$.*

Then $\overline{F(X)} \cap K \cap \bigcap\{G(x) \mid x \in D\} \neq \emptyset$.

For $X = Y$ and $F = 1_X$, condition (2) simply states that $G : D \multimap X$ is a KKM map. In [18] we showed that many of the key results of a large number of other papers are consequences of the preceding theorem.

(II) In 1994 [21], we introduced an *admissible* class $\mathfrak{A}_c^\kappa(X, Y)$ of maps $T : X \multimap Y$ between topological spaces X and Y as the one such that, for each T and each compact subset K of X, there exists a map $\Gamma \in \mathfrak{A}_c(K, Y)$ satisfying $\Gamma(x) \subset T(x)$ for all $x \in K$; where $\mathfrak{A}_c$ is consisting of finite composites of maps in $\mathfrak{A}$, and $\mathfrak{A}$ is a class of maps satisfying the following properties:
(i) $\mathfrak{A}$ contains the class $\mathbb{C}$ of (single-valued) continuous functions;
(ii) each $F \in \mathfrak{A}_c$ is u.s.c. and compact-valued; and
(iii) for any polytope P, each $F \in \mathfrak{A}_c(P, P)$ has a fixed point.

Theorem. ([21]) *Let D be a nonempty subset of a convex space X, Y a Hausdorff space, and $F \in \mathfrak{A}_c^\kappa(X, Y)$. Let $G : D \multimap Y$ be a multimap such that*
(1) *for each $x \in D$, $G(x)$ is closed in Y;*
(2) *for any $N \in \langle D\rangle$, $F(\mathrm{co}\, N) \subset G(N)$; and*
(3) *there exist a nonempty compact subset K of Y and, for each $N \in \langle D\rangle$, a compact D-convex subset L_N of X containing N such that $F(L_N) \cap \bigcap\{Gx \mid x \in L_N \cap D\} \subset K$.*
Then $\overline{F(X)} \cap K \cap \bigcap\{Gx \mid x \in D\} \neq \emptyset$.

Note that, if F is single-valued, the Hausdorffness assumption on Y is not necessary.

(III) The origin of the class $\mathfrak{KC}$ and Theorem C is the following [47, Theorem 3]:

Theorem. ([47]) *Let $(X \supset D; \Gamma)$ be a G-convex space, Y a Hausdorff space, and $F \in \mathfrak{A}_c^\kappa(X, Y)$. Let $G : D \multimap Y$ be a map such that*
(1) *for each $x \in D$, $G(x)$ is closed in Y;*
(2) *for any $N \in \langle D\rangle$, $F(\Gamma_N) \subset G(N)$; and*
(3) *there exist a nonempty compact subset K of Y such that either*
(i) *$\bigcap\{G(x) \mid x \in M\} \subset K$ for some $M \in \langle D\rangle$; or*
(ii) *for each $N \in \langle D\rangle$, a compact G-convex subset L_N of X containing N such that $F(L_N) \cap \bigcap\{G(x) \mid x \in L_N \cap D\} \subset K$.*
Then $\overline{F(X)} \cap K \cap \bigcap\{G(x) \mid x \in D\} \neq \emptyset$.

This was due to Park and Kim [47,48], where this had been reformulated to more than a dozen foundational results in the KKM theory. The class $\mathfrak{A}_c^\kappa$ in the above theorem can be replaced by the extended class $\mathfrak{B}$ for G-convex spaces.

This was given originally under the assumption that $D \subset X$, which is redundant in view of condition (ii) of Theorem C. In the preceding theorem, the admissible class $\mathfrak{A}_c^\kappa$ is a subclass of $\mathfrak{KC}$; and note that $F(L_N)$ is compact since L_N is compact and F can be regarded as a composition of u.s.c. maps having compact values (by the definition of $\mathfrak{A}_c^\kappa$).

In 1997 [48], we gave ten equivalent formulation of the preceding theorem in the form of coincidence theorems, matching theorems, analytic alternatives, minimax inequalities, geometric and section properties. Similarly, we can make equivalent formulations of Theorem C. Moreover, in [48], a large number of particular forms of the preceding theorem are listed. Note also that, after [48], there have appeared too many similar works on G-convex spaces and modifications of the preceding theorem to trace out all of them.

(IV) The KKM theorem in (III) was modified by Kalmoun and Rihai [5] in 2001 as follows: For a transfer closed-valued map G in (1) and by considering $\overline{G}$ instead of G in (2) and (3), they deduced the same conclusion as in Theorem C. By applying this modification, they obtained an existence theorem for generalized vector equilibrium problems and applied it to greatest element, fixed point, and vector saddle point problems within the frame of G-convex spaces. A particular form of their KKM theorem was applied in 2003 [6] to existence for vector equilibrium, mixed variational inequalities, greatest elements for a binary relation, and the Fan-Browder fixed point theorem.

(V) Let X be a convex space and Y a Hausdorff space. In 1997 [22], we introduced a new "better" admissible class $\mathfrak{B}$ of multimaps as follows:

$F \in \mathfrak{B}(X,Y) \iff F : X \multimap Y$ such that, for any polytope P in X and any continuous map $f : F(P) \longrightarrow P$, $f(F|_P)$ has a fixed point.

The following KKM theorem is due to the author [22, Theorem 3]:

Theorem. ([22]) *Let X be a convex space, Y a Hausdorff space, $F \in \mathfrak{B}(X,Y)$ a compact map, and $S : X \multimap Y$ a map. Suppose that*
(1) *for each $x \in X$, $S(x)$ is closed; and*
(2) *for each $N \in \langle X \rangle$, $F(\operatorname{co} N) \subset S(N)$.*
Then $\overline{F(X)} \cap \bigcap\{S(x) \mid x \in X\} \neq \emptyset$.

This KKM theorem was applied in [23] to a minimax inequality related to admissible multimaps, from which we deduced generalized versions of lopsided saddle point theorems, fixed point theorems, existence of maximizable linear functionals, the Warlas excess demand theorem, and the Gale-Nikaido-Debreu theorem.

In 2010 [1], the preceding theorem is shown to be equivalent to some existence theorems of variational inclusion problems. These were applied to existence theorems of common fixed point, generalized maximal element theorems, a general coincidence theorems and a section theorem.

(VI) There have also appeared a large number of the so-called generalized KKM maps in the literature. In fact, a number of authors tried to generalize the concept of KKM maps on particular cases of ϕ_A-spaces. All such KKM maps are known to be the ones for certain G-convex spaces; see [34].

6. COMMENTS ON RELATED WORKS

There are several hundred published papers on generalizations of the KKM theorem. Recently, in order to upgrade the KKM theory, we have tried to criticize some inappropriate results of other authors. We give abstracts of a few examples of such papers by the present author:

(I) In [34], we introduced basic results in the KKM theory on abstract convex spaces and the KKM maps. These were applied to various modifications of the concepts of generalized convex spaces and KKM type maps. We discuss the nature of those modifications and criticize recently appeared 'generalizations' of our previous works due to many other authors.

(II) Basic results in the KKM theory on abstract convex spaces and the KKM maps are applied to ϕ_A-spaces which unify various imitations of G-convex spaces in [37]. We also showed that basic theorems on ϕ_A-spaces can be applied to correct and improve results on the so-called R-KKM maps on the so-called L-convex spaces in a work of C.M. Chen.

(III) In [38], we introduced a new concept of abstract convex minimal spaces which was used to establish typical results in the KKM theory. Since any minimal space can be made into a topological space, results on abstract convex minimal spaces can be deduced from the theory on abstract convex spaces. In this way, the KKM type theorems were used to obtain coincidence theorems, the Fan-Browder type fixed point theorems, the Fan intersection theorem, and the Nash equilibrium theorem on abstract convex minimal spaces.

(IV) Recently, some authors adopted the concept of the so-called *generalized R-KKM maps* which were used to rewrite known results in the KKM theory. In [44], we showed that those maps are simply KKM maps on G-convex spaces. Consequently, results on generalized R-KKM maps follow the corresponding previous ones on G-convex spaces.

(V) In a paper by Hou Jicheng, *On some KKM type theorems,* Advances in Mathematics, 36(1) (2007), 86–88, the author claimed that some previous KKM type theorems are false by giving a counterexample. In [45], we showed that the counterexample does not work and, consequently, the results are correct. Moreover, we claimed that the artificial concept like transfer compactly closed-valued maps can be destroyed. Finally, we introduced a theorem generalizing the main target of Hou.

(VI) In the KKM theory, instead of the concepts of closure, interior, closed-valued multimap, l.s.c. multimap, finite open cover, etc., resp., some authors adopt ccl, cint, transfer compactly closed-valued map, transfer compactly l.s.c. multimap, transfer compactly local intersection property, etc., resp. In [46], we showed that such inappropriate and artificial concepts can be invalidated. For example, by giving finer topologies on the underlying space, we can invalidate “compactly"-attached terminology. In such ways, we obtained simpler formulations of some KKM type theorems, some Fan-Browder type fixed point theorems, and others.

This is why we eliminated such useless terminology in this paper.

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References

1. M. Balaj, L.-J. Lin, *Equivalent forms of a generalized KKM theorem and their applications*, Nonlinear Anal. TMA **73** (2010), 673–682.
2. S.-Y. Chang, *A generalization of KKM principle and its applications*, Soochow J. Math. **15** (1989), 7–17.
3. K. Fan, *A generalization of Tychonoff's fixed point theorem*, Math. Ann. **142** (1961), 305–310.

4. K. Fan, *Some properties of convex sets related to fixed point theorems*, Math. Ann. **266** (1984), 519–537.
5. E. M. Kalmoun, H. Riahi, *Topological KKM theorems and generalized vector equilibria on G-convex spaces with applications*, Proc. Amer. Math. Soc. **129**(5), 1335–1348.
6. E. M. Kalmoun, *On some new KKM theorems and related applications*, J. Nonlinear Funct. Anal. Appl. **8**(1) (2003), 1–23.
7. H. Kim, S. Park, *Generalized KKM maps, maximal elements, and almost fixed points*, J. Korean Math. Soc. **44**(2) (2007), 393–406.
8. W. K. Kim, *Some applications of the Kakutani fixed point theorem*, J. Math. Anal. Appl. **121** (1987), 119–122.
9. W. K. Kim, *Some intersection theorems of the KKM-maps*, Bull. Korean Math. Soc. **24** (1987), 139–144.
10. B. Knaster, K. Kuratowski, S. Mazurkiewicz, *Ein Beweis des Fixpunktsatzes für n-Dimensionale Simplexe*, Fund. Math. **14** (1929), 132–137.
11. M. Lassonde, *On the use of KKM multifunctions in fixed point theory and related topics*, J. Math. Anal. Appl. **97** (1983), 151–201.
12. M. Lassonde, *Sur le principe KKM*, C. R. Acad. Sci. Paris **310** (1990), 573–576.
13. D. T. Luc, E. Sarabi, A. Soubeyran, *Existence of solutions in variational relation problems without convexity*, J. Math. Anal. Appl. **364** (2010), 544–555.
14. S. Park, *Generalized Fan-Browder fixed point theorems and their applications*, Collec. of Papers Dedicated to J. G. Park, Chonbuk National University, 1989, pp.51–77.
15. S. Park, *Generalizations of Ky Fan's matching theorems and their applications*, J. Math. Anal. Appl. **141** (1989), 164–176.
16. S. Park, *Convex spaces and KKM families of subsets*, Bull. Korean Math. Soc. **27** (1990), 11-14.
17. S. Park, *Some coincidence theorems on acyclic multifunctions and applications to KKM theory*, in: K.-K. Tan (Ed.), Fixed Point Theory and Applications, World Sci. Publ., River Edge, NJ, 1992, pp.248–277.
18. S. Park, *On the KKM type theorems on spaces having certain contractible subsets*, Kyungpook Math. J. **32** (1992), 607–628.
19. S. Park, *On minimax inequalities on spaces having certain contractible subsets*, Bull. Austral. Math. Soc. **47** (1993), 25–40.
20. S. Park, *A unified approach to generalizations of the KKM type theorems related to acyclic maps*, Numer. Funct. Anal. Optimiz. **15** (1994), 105–119.
21. S. Park, *Foundations of the KKM theory via coincidences of composites of admissible u.s.c. maps*, J. Korean Math. Soc. **31** (1994), 493-516.
22. S. Park, *Coincidence theorems for the better admissible multimaps and their applications*, Nonlinear Anal. TMA **30**(7) (1997), 4183–4191.
23. S. Park, *A generalized minimax inequality related to admissible multimaps and its applications*, J. Korean Math. Soc. **34** (1997), 719–730.
24. S. Park, *Ninety years of the Brouwer fixed point theorem*, Vietnam J. Math. **27** (1999), 193–232.
25. S. Park, *Elements of the KKM theory for generalized convex spaces*, Korean J. Comput. Appl. Math. **7** (2000), 1–28.
26. S. Park, *Remarks on topologies of generalized convex spaces*, Nonlinear Funct. Anal. Appl. **5** (2000), 67–79.
27. S. Park, *On generalizations of the KKM principle on abstract convex spaces*, Nonlinear Anal. Forum **11** (2006), 67–77.

28. S. Park, *Various subclasses of abstract convex spaces for the KKM theory*, Proc. Nat. Inst. Math. Sci. **2**(4) (2007), 35–47.
29. S. Park, *Remarks on $\mathfrak{KC}$-maps and $\mathfrak{KO}$-maps in abstract convex spaces*, Nonlinear Anal. Forum **12**(1) (2007), 29–40.
30. S. Park, *Comments on some abstract convex spaces and the KKM maps*, Nonlinear Anal. Forum, **12**(2) (2007), 125–139.
31. S. Park, *Elements of the KKM theory on abstract convex spaces*, J. Korean Math. Soc. **45**(1) (2008), 1–27.
32. S. Park, *Equilibrium existence theorems in KKM spaces*, Nonlinear Anal. TMA **69** (2008), 4352–4364.
33. S. Park, *New foundations of the KKM theory*, J. Nonlinear Convex Anal. **9**(3) (2008), 331–350.
34. S. Park, *Comments on recent studies on abstract convex spaces*, Nonlinear Anal. Forum **13**(1) (2008), 1–17.
35. S. Park, *Remarks on fixed points, maximal elements, and equilibria of economies in abstract convex spaces*, Taiwanese J. Math. **12**(6) (2008), 1365–1383.
36. S. Park, *Remarks on KKM maps and fixed point theorems in generalized convex spaces*, CUBO, Math. J. **10**(4) (2008), 1–13.
37. S. Park, *Comments on the KKM theory on ϕ_A-spaces*, PanAmerican Math. J. **18**(2) (2008), 61–71.
38. S. Park, *Applications of the KKM principle on abstract convex minimal spaces*, Nonlinear Funct. Anal. Appl. **13**(2) (2008), 179–191.
39. S. Park, *Remarks on the partial KKM principle*, Nonlinear Anal. Forum **14** (2009), 51–62.
40. S. Park, *Fixed point theory of multimaps in abstract convex uniform spaces*, Nonlinear Anal. TMA **71** (2009), 2468–2480.
41. S. Park, *Generalized convex spaces, L-spaces, and FC-spaces*, J. Global Optim. **45** (2009), 203–210.
42. S. Park, *General KKM theorems for abstract convex spaces*, J. Inf. Math. Sci. **1**(1) (2009), 1–13.
43. S. Park, *The KKM principle in abstract convex spaces: Equivalent formulations and applications*, Nonlinear Anal. TMA **73** (2010), 1028–1042.
44. S. Park, *Comments on generalized R-KKM type theorems*, Comm. Korean Math. Soc. **25**(2) (2010), 303–311.
45. S. Park, *Comments on Hou Jicheng's "On Some KKM Type Theorems"*, Comm. Korean Math. Soc. **25**(3) (2010), 491–495.
46. S. Park, *Remarks on basic concepts in the KKM theory*, Nonlinear Anal. TMA **74** (2011), 2349-2447.
47. S. Park, H. Kim, *Coincidence theorems for admissible multifunctions on generalized convex spaces*, J. Math. Anal. Appl. **197** (1996), 173–187.
48. S. Park, H. Kim, *Foundations of the KKM theory on generalized convex spaces*, J. Math. Anal. Appl. **209** (1997), 551–571.
49. S. Park, W. Lee, *A unified approach to generalized KKM maps in generalized convex spaces,* J. Nonlinear Convex Anal. **2** (2001), 157–166.
50. R.T. Rockafellar, *Convex Analysis*, Princeton Univ. Press, Princeton, NJ, 1970.
51. M.-H. Shih, K.-K. Tan, *Covering theorems of convex sets related to fixed-point theorems*, in: B.-L. Lin, S. Simons (Eds.), Nonlinear and Convex Analysis—Proc. in Honor of Ky Fan, Marcel Dekker, Inc., New York, 1987, pp.235–244.