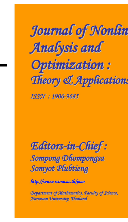
Journal of Nonlinear Analysis and Optimization
Vol. 2, No. 1, (2011), 239-248
ISSN : 1906-9605
http://www.sci.nu.ac.th/jnao



ON STATES WITH ABSORBING TENDENCIES IN SELF-ORGANIZING MAPS WITH INPUTS IN AN INNER PRODUCT SPACE

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ABSTRACT. We deal with self-organizing map models referred to as Kohonen type algorithm. In self-organizing maps, it is easy to observe some practical and interesting properties in the relation between the arrangement of the nodes and their values. We shall be concerned with behavior of ordering, state preserving properties and tendencies to be absorbed to a particular state in self-organizing maps with inputs taking values in an inner product space. We give a numerical example as its application and estimate ordering by simulation approach.

KEYWORDS : Self-organizing maps; Absorbing states.

MSC : 68T05.

1. Formulation of self-organizing maps

We consider self-organizing map models referred to as Kohonen [8] type algorithm. Self-organizing map algorithm is very practical and has many useful applications, semantic map, diagnosis of speech voicing, solving traveling-salesman problem, and so on. There are some interesting phenomena between the array of nodes and the values of nodes in these models. Indeed practical properties in self-organizing map models are easy to observe, but they still remain

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Article history : Received 28 January 2010. Accepted 1 February 2011.

without mathematical proof in general cases. Concerning the mathematical analysis of self-organizing maps, a proof of the convergence of the learning process in the one-dimensional case was first given by Cottrell and Fort [1] and convergence properties are more generally studied, e.g., in Erwin, Obermayer, and Schulten [2][3][4]. The purpose of this paper is to make a study of behavior of ordering and tendencies to be absorbed to a particular state in self-organizing maps with inputs taking values in an inner product space.

We consider to characterize a model $(I, V, X, \{m_k(\cdot)\}_{k=0}^{\infty})$ with four elements which consist of the *nodes*, the *values of nodes*, *inputs* and *model functions* with some *learning processes*, in this paper. There are several types of models with various spaces of nodes, spaces of their values and ways of learning for nodes. We suppose the followings.

(i) We suppose an array of *nodes*. Let I denote the set of all nodes, which is called the *node set*. We assume that I is a countable set metrized by a metric d. In many applications, usually, we use the following ones, a finite subset of the set $\mathbb{N}$ of all natural numbers, or a finite subset of $\mathbb{N}^2$.

(ii) We suppose that each node has its *value*. V is the space of values of nodes. We assume that V is a normed linear space with $\|\cdot\|$. A mapping $m : I \longrightarrow V$ transforming each node i to its value $m(i)$ is called a *model function*. Let M be the set of all model functions.

(iii) X is the *input set*. Let X be a subset of V. $x \in X$ is called an *input*.

(iv) The *learning process* is defined by the following. If an input is given, then the value of each node is renewed to a new value according to the input. If an input x is given, node i learns x and its value $m(i)$ changes to a new value $m'(i)$ determined by

$$m'(i) = (1 - \alpha_{m,x}(i))m(i) + \alpha_{m,x}(i)x \tag{1.1}$$

according to the rate $\alpha_{m,x}(i) \in [0, 1]$. If an initial model function m_0 and a sequence $x_0, x_1, x_2, \ldots \in X$ of inputs are given, then the model functions $m_1, m_2, m_3, \ldots$ are generated sequentially according to the following equation.

$$m_{k+1}(i) = (1 - \alpha_{m_k,x_k}(i))m_k(i) + \alpha_{m_k,x_k}(i)x_k. \tag{1.2}$$

There are several types of models with various spaces of nodes, spaces of their values and ways of learning for nodes. In this paper, we use two types of learning processes defined by the following.

Learning process $\mathrm{L_A}$:

(i) Areas of learning: for each $m_k \in M$ and $x_k \in X$,

$$I(m_k, x_k) = \left\{ i^* \in I \;\middle|\; \|m_k(i^*) - x_k\| = \inf_{i \in I} \|m_k(i) - x_k\| \right\},$$
$$N_\varepsilon(i) = \left\{ j \in I \;\middle|\; d(j,i) \le \varepsilon \right\} \quad (\varepsilon > 0 \text{ is the learning radius, } i \in I).$$

(ii) Learning-rate factor: $0 \le \alpha \le 1$.

(iii) Learning:

$$m_{k+1}(i) = \begin{cases} (1-\alpha)m_k(i) + \alpha x_k & \text{if } i \in \bigcup_{i^* \in I(m_k,x_k)} N_\varepsilon(i^*), \\ m_k(i) & \text{if } i \notin \bigcup_{i^* \in I(m_k,x_k)} N_\varepsilon(i^*), \end{cases} \quad k = 0, 1, 2, \ldots.$$

We note that, if $I(m_k, x_k) = \varnothing$ then $m_{k+1} = m_k$.

Learning process $\mathrm{L_m}$: This learning process is the same as Learning process $\mathrm{L_A}$ except that a node $J(m_k, x_k)$ is selected from $I(m_k, x_k)$ by a given rule. Let $J : M \times X \longrightarrow I$ be a mapping which satisfies $J(m_k, x_k) \in I(m_k, x_k)$. For example, if I is a totally ordered finite set, $J(m_k, x_k)$ may be defined by $J(m_k, x_k) = \min I(m_k, x_k)$. If $i \in N_\varepsilon(J(m_k, x_k))$ then $m_{k+1}(i) = (1-\alpha)m_k(i) + \alpha x_k$, otherwise $m_{k+1}(i) = m_k(i)$. We assume that $m_{k+1} = m_k$ if $I(m_k, x_k) = \varnothing$.

2. A fundamental self-organizing map and absorbing states

In Section 2, we restrict our considerations to basic self-organizing maps with real-valued nodes and one dimensional array of nodes. Now, we suppose that set V of values of nodes is identified with $\mathbb{R}$ which is the set of all real numbers.

$$(I = \{1, 2, \ldots, n\}, V = \mathbb{R}, X \subset \mathbb{R}, \{m_k(\cdot)\}_{k=0}^{\infty})$$

(i) Let $I = \{1, 2, \ldots, n\}$ be the node set with metric $d(i,j) = |i - j|$. (ii) Assume $V = \mathbb{R}$, that is, each node is $\mathbb{R}$-valued. A model function m is written as $m = [m(1), m(2), \ldots, m(n)]$. (iii) $x_0, x_1, x_2, \ldots \in X \subset \mathbb{R}$ is an input sequence. (iv) Learning process $\mathrm{L_A}$ (1-dimensional array, $\mathbb{R}$-valued nodes and $\varepsilon = 1$): (a) areas of learning: $I(m_k, x_k) = \{i^* \in I \mid |m_k(i^*) - x_k| = \inf_{i \in I} |m_k(i) - x_k|\}$ and $N_1(i) = \{j \in I \mid |j - i| \le 1\}$; (b) learning-rate factor: $0 \le \alpha \le 1$; (c) learning: for each $k = 0, 1, 2, \ldots$, if $i \in \cup_{i^* \in I(m_k, x_k)} N_1(i^*)$ then $m_{k+1}(i) = (1-\alpha)m_k(i) + \alpha x_k$, otherwise $m_{k+1}(i) = m_k(i)$.

If an input $x_0 \in X$ is given, then we choose node i^* which has the most similar value to x_0 within $m_0(1), m_0(2), \ldots, m_0(n)$. Node i^* and the nodes which are in the neighborhood of i^* learn x_0 and their values change to new values $m_1(i) = (1-\alpha)m_0(i) + \alpha x_0$. The nodes which are not in the neighborhood of i^* do not learn and their values do not change. Repeating these updating for the inputs $x_1, x_2, x_3, \ldots$, the value of each node is renewed sequentially. Simultaneously, model functions $m_1, m_2, m_3, \ldots$ are also generated sequentially.

The following properties are well-known results and can be verified easily.

Theorem 2.1. *We consider a self-organizing map model* $(I = \{1, 2, \ldots, n\}$, $V = \mathbb{R}, X \subset \mathbb{R}, \{m_k(\cdot)\}_{k=0}^{\infty})$ *with Learning process* $L_A(\varepsilon = 1)$*. For model functions* m_1, m_2, m_3, $\ldots$, *the following statements hold:*

(i) *if* m_k *is increasing on* I*, then* m_{k+1} *is increasing on* I*;*
(ii) *if* m_k *is decreasing on* I*, then* m_{k+1} *is decreasing on* I*;*
(iii) *if* m_k *is strictly increasing on* I*, then* m_{k+1} *is strictly increasing on* I*;*
(iv) *if* m_k *is strictly decreasing on* I*, then* m_{k+1} *is strictly decreasing on* I*.*

Such properties as monotonicity are called *absorbing states* of self-organizing map models in the sense that once model function leads to increasing state, it never leads to other states for the learning by any input. Quasi-convexity or quasi-concavity is also an absorbing state of the previous self-organizing maps. Their details and definitions of quasi-convexity and quasi-concavity for model functions are in [6].

3. States with absorbing tendencies in 2-dimensional array model

In this section, we suppose the case of nodes with values in a normed linear space and 2-dimensional array.

$$\big(\{1, 2, \ldots, n_1\} \times \{1, 2, \ldots, n_2\},\ V,\ X,\ \{m_k(\cdot,\cdot)\}_{k=0}^{\infty}\big)$$

(i) The node set. Let $I = \{1, 2, \ldots, n_1\} \times \{1, 2, \ldots, n_2\}$ with metric

$$d_I((i,j),(k,l)) = \sqrt{(i-k)^2 + (j-k)^2}, \quad (i,j),(k,l) \in I.$$

(ii) The values of nodes. Let $m : I \longrightarrow V$, where V is a normed linear space with an inner product $\langle \cdot, \cdot \rangle$.
(iii) $x_0, x_1, x_2, \ldots \in X \subset V$ is an input sequence.
(iv) Assume Larning process L_m (2-dimensional array and $\varepsilon = \sqrt{2}$)
  (a) areas of learning:

$$I(m,x) = \{(i^*, j^*) \in I \mid \|m(i^*, j^*) - x\| = \inf_{(i,j)\in I} \|m(i,j) - x\|\},$$

  $m \in M, x \in X$, let $J : M \times X \longrightarrow I$ be a mapping which satisfies $J(m,x) \in I(m,x)$, and

$$N_{\sqrt{2}}(i,j) = \{(k,l) \in I \mid d_I((i,j),(k,l)) \le \sqrt{2}\};$$

  (b) learning-rate factor: $0 \le \alpha \le 1$;
  (c) learning: if $(i,j) \in N_{\sqrt{2}}(J(m,x))$ then $m'(i,j) = (1 - \alpha)m(i,j) + \alpha x$, otherwise $m'(i,j) = m(i,j)$.

The following property can be use as measures of ordering and analysis of pre-ordered state in the learning processes.

Theorem 3.1. *We consider a self-organizing map model*

$$\big(\{1,2,\ldots,n_1\}\times\{1,2,\ldots,n_2\},\ V,\ X,\ \{m_k(\cdot,\cdot)\}_{k=0}^{\infty}\big)$$

with Learning process $L_m(\varepsilon=\sqrt{2})$*. Let* m *be an arbitrary model function and* x *an arbitrary input. Let* m' *be the renewed model function of* m *by* x*. For every node* (i,j) *with* $d_I((i,j),J(m,x))\neq\sqrt{2},2,\sqrt{5}$*, if*

$$\langle m(i-1,j)-m(i,j), m(i+1,j)-m(i,j)\rangle\leq 0 \tag{3.1}$$

and

$$\langle m(i,j-1)-m(i,j), m(i,j+1)-m(i,j)\rangle\leq 0 \tag{3.2}$$

hold, then

$$\langle m'(i-1,j)-m'(i,j), m'(i+1,j)-m'(i,j)\rangle\leq 0 \tag{3.3}$$

and

$$\langle m'(i,j-1)-m'(i,j), m'(i,j+1)-m'(i,j)\rangle\leq 0 \tag{3.4}$$

hold.

Proof. We note that $d_I((i,j),J(m,x))=0,1,\sqrt{2},2,\sqrt{5},2\sqrt{2},\ldots$.

(A) For $d_I((i,j),J(m,x))\geq 2\sqrt{2}$, we have

$$m'(k,l)=m(k,l),\quad (k,l)=(i-1,j),(i,j),(i+1,j),(i,j-1),(i,j+1).$$

Therefore, (3.1) implies (3.3) and (3.2) implies (3.4).

(B) For $d_I((i,j),J(m,x))=0$, we have

$$m'(k,l)=(1-\alpha)m(k,l)+\alpha x,$$
$$(k,l)=(i-1,j),(i,j),(i+1,j),(i,j-1),(i,j+1).$$

Therefore, if (3.1) holds, then

$$\begin{aligned} & \langle m'(i-1,j)-m'(i,j), m'(i+1,j)-m'(i,j)\rangle \\ = & \langle (1-\alpha)m(i-1,j)+\alpha x-(1-\alpha)m(i,j)-\alpha x, \\ & (1-\alpha)m(i+1,j)+\alpha x-(1-\alpha)m(i,j)-\alpha x\rangle \\ = & (1-\alpha)^2\langle m(i-1,j)-m(i,j), m(i+1,j)-m(i,j)\rangle \\ \leq & \ 0. \end{aligned}$$

Similarly, (3.2) implies (3.4).

($C_{1,0}$) For $(i,j)=J(m,x)+(1,0)$, we have

$$m'(k,l)=\begin{cases}(1-\alpha)m(k,l)+\alpha x, & \text{if } (k,l)=(i-1,j),(i,j),(i,j-1),(i,j+1),\\ m(k,l)+\alpha x, & \text{if } (k,l)=(i+1,j).\end{cases}$$

Therefore

$$\langle m'(i-1,j)-m'(i,j), m'(i+1,j)-m'(i,j)\rangle$$

$$\begin{aligned} &= \langle (1-\alpha)m(i-1,j)+\alpha x-(1-\alpha)m(i,j)-\alpha x,\\ &\quad m(i+1,j)-(1-\alpha)m(i,j)-\alpha x\rangle\\ &= (1-\alpha)\langle m(i-1,j)-m(i,j), m(i+1,j)-m(i,j)+\alpha(m(i,j)-x)\rangle\\ &= (1-\alpha)\langle m(i-1,j)-m(i,j), m(i+1,j)-m(i,j)\rangle\\ &\quad -\alpha(1-\alpha)\langle m(i-1,j)-m(i,j), x-m(i,j)\rangle\\ &= (1-\alpha)\langle m(i-1,j)-m(i,j), m(i+1,j)-m(i,j)\rangle\\ &\quad -\frac{1}{2}\alpha(1-\alpha)\{\|m(i-1,j)-m(i,j)\|^2+\|m(i,j)-x\|^2\\ &\quad -\|m(i-1,j)-x\|^2\}. \end{aligned}$$

Since $(i-1,j)=J(m,x)$, we have $\|m(i-1,j)-x\| \le \|m(i,j)-x\|$. It follows that

$$\begin{aligned} &\langle m'(i-1,j)-m'(i,j), m'(i+1,j)-m'(i,j)\rangle\\ \le\ &\langle m(i-1,j)-m(i,j), m(i+1,j)-m(i,j)\rangle. \end{aligned}$$

Hence, if (3.1) holds, then (3.3) holds. Since

$$\begin{aligned} &\langle m'(i,j-1)-m'(i,j), m'(i,j+1)-m'(i,j)\rangle\\ \le\ &\langle m(i,j-1)-m(i,j), m(i,j+1)-m(i,j)\rangle, \end{aligned}$$

(3.2) implies (3.4).

($C_{0,1}$) For $(i,j)=J(m,x)+(0,1)$, we have $m'(k,l)=$

$$\begin{cases} (1-\alpha)m(k,l)+\alpha x, & \text{if } (k,l)=(i-1,j),(i,j),(i+1,j),(i,j-1),\\ m(k,l)+\alpha x, & \text{if } (k,l)=(i,j+1). \end{cases}$$

By the same argument used in ($C_{1,0}$),

$$\begin{aligned} &\langle m'(i,j-1)-m'(i,j), m'(i,j+1)-m'(i,j)\rangle\\ =\ &(1-\alpha)\langle m(i,j-1)-m(i,j), m(i,j+1)-m(i,j)\rangle\\ &-\frac{1}{2}\alpha(1-\alpha)\big\{\|m(i,j-1)-m(i,j)\|^2+\|m(i,j)-x\|^2\\ &-\|m(i,j-1)-x\|^2\big\}. \end{aligned}$$

Since $(i,j-1)=J(m,x)$, we have $\|m(i,j-1)-x\| \le \|m(i,j)-x\|$. Hence, if (3.2) holds, then (3.4) holds. Since the left hand side of (3.1) equals the left hand side of (3.3) for $(i,j)=J(m,x)+(0,1)$, (3.1) implies (3.3).

($C_{-1,0}$) For $(i,j)=J(m,x)+(-1,0)$, $m'(k,l)=$

$$\begin{cases} (1-\alpha)m(k,l)+\alpha x, & \text{if } (k,l)=(i,j),(i+1,j),(i,j-1),(i,j+1),\\ m(k,l)+\alpha x, & \text{if } (k,l)=(i-1,j). \end{cases}$$

Then, we have

$$\begin{aligned} & \langle m'(i-1,j) - m'(i,j), m'(i+1,j) - m'(i,j)\rangle \\ = \;& (1-\alpha)\langle m(i-1,j) - m(i,j), m(i+1,j) - m(i,j)\rangle \\ & -\frac{1}{2}\alpha(1-\alpha)\big\{\|x - m(i,j)\|^2 + \|m(i+1,j) - m(i,j)\|^2 \\ & -\|x - m(i+1,j)\|^2\big\}. \end{aligned}$$

Since $(i+1,j) = J(m,x)$, we have $\|x - m(i+1,j)\| \leq \|x - m(i,j)\|$. Hence, if (3.1) holds, then (3.3) holds. Similarly, (3.2) implies (3.4) for $(i,j) = J(m,x) + (-1,0)$.

($C_{0,-1}$) For $(i,j) = J(m,x) + (0,-1)$, similarly, (3.3) and (3.4) hold. □

4. 1-DIMENSIONAL ARRAY CASE AND A NUMERICAL EXAMPLE

In this section, we suppose the case of nodes with values in a normed linear space and 1-dimensional array.

$$\big(\{1,2,\dots,n\},\ V,\ X,\ \{m_k(\cdot)\}_{k=0}^{\infty}\big)$$

(i) The node set. Let $I = \{1,2,\dots,n\}$ with metric $d_I(i,j) = |i-j|$, $i,j \in I$.

(ii) The values of nodes. Let $m : I \longrightarrow V$, where V is a normed linear space with an inner product $\langle\cdot,\cdot\rangle$.

(iii) $x_0, x_1, x_2, \dots \in X \subset V$ is an input sequence.

(iv) Assume Larning process L_m (1-dimensional array and $\varepsilon = 1$)

(a) areas of learning:

$$J(m,x) = \min\{i^* \in I \mid \|m(i^*) - x\| = \inf_{i\in I}\|m(i) - x\|\}, \quad m \in M, x \in X$$

and $N_1(i) = \{j \in I \mid d_I(i,j) \leq 1\}$;

(b) learning-rate factor: $0 \leq \alpha \leq 1$;

(c) learning: if $i \in N_1(J(m,x))$ then $m'(i) = (1-\alpha)m(i) + \alpha x$, otherwise $m'(i) = m(i)$.

For 1-dimensional array case, we have the following property and it is proved by the similar argument in the proof of Theorem 3.1.

Theorem 4.1. *We consider a self-organizing map model*

$$\big(\{1,2,\dots,n\},\ V,\ X,\ \{m_k(\cdot)\}_{k=0}^{\infty}\big)$$

with Learning process $L_m(\varepsilon = 1)$. Let m be an arbitrary model function and x an arbitrary input. Let m' be the renewed model function of m by x. For every node i with $d_I(i, J(m,x)) \neq 2$, if

$$\langle m(i-1) - m(i), m(i+1) - m(i)\rangle \leq 0 \tag{4.1}$$

hold, then

$$\langle m'(i-1) - m'(i), m'(i+1) - m'(i)\rangle \leq 0$$

hold. Particularly, in case $0 \leq \alpha < 1$, *for every node* i *with* $d_I(i, J(m,x)) \neq 2$, *if*

$$\langle m(i-1) - m(i), m(i+1) - m(i) \rangle < 0$$

hold, then

$$\langle m'(i-1) - m'(i), m'(i+1) - m'(i) \rangle < 0$$

hold.

We give a numerical example of 1-dimensional array model with 2-dimensional inputs. We assume the following.

(i) Let $I = \{1, 2, 3, \ldots, 35\}$ be the node set with metric $d_I(i,j) = |i - j|$.

(ii) Initial model function is given by $m_0(1) = (3,1), m_0(2) = (9,8), \ldots, m_0(35) = (0,7)$ (see Figure 2). For $x = (x_1, x_2), y = (y_1, y_2)$, $\|x - y\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$.

(iii) Suppose 19000 inputs as follows, randomly, generated by the distribution described in Figure 1. For example, the distribution in Figure 1 means a distribution map of a certain population, as demand. A densely distributed area has a large population.

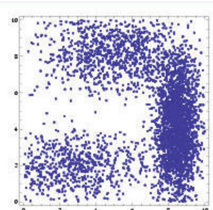
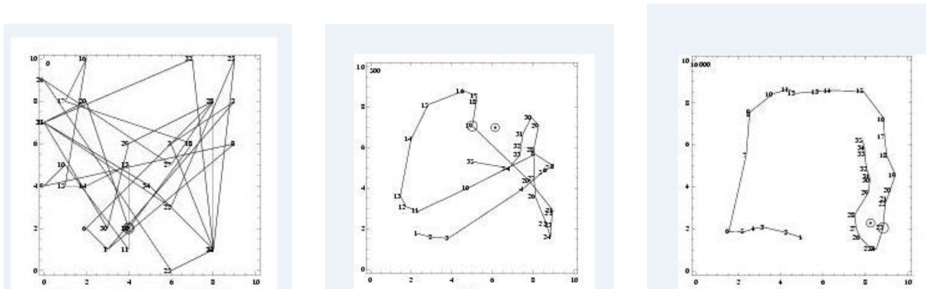


FIGURE 1. Distribution of inputs as demands

(iv) Assume Learning process $L_m(\varepsilon = 1)$ with learning-rate factor $\alpha = \frac{1}{3}$.

Figure 2 illustrates nodes and their values in each iteration steps. The position of every node means its value. The length of a path is defined by $\sum_{i=1}^{n-1} \|m(i) - m(i+1)\|$. By repeating learning, we can observe that the values of the nodes in each iteration step are ordered and that their values yield a well-balanced solution with a shorter path and satisfying demands. The path constructed by model function m_{16000} has more nodes in densely distributed areas on Figure 1.

Figure 3 shows relative frequencies of nodes satisfy condition (4.1) defined by



The length of path 182.92 43.28
29.01

FIGURE 2. Initial values of nodes (m_0), 500 renewals (m_{500}) and 16000 renewals (m_{16000})

$$r_{\text{inn}} = \frac{\text{the number of elements in } \{i \mid \langle m(i-1) - m(i), m(i+1) - m(i) \rangle \leq 0\}}{n-2},$$

where n is the number of nodes. This rate can be use as measures of ordering and estimations of convergence in the learning processes.

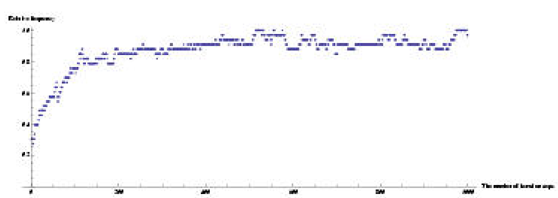


FIGURE 3. Transition of relative frequencies r_{inn}

Figures 4 and 5 show transitions of the length of path and the mean distance between a model function m and inputs defined by

$$\text{MD} = \frac{\sum_{x \in X} \min_{i \in I} \|m(i) - x\|}{N(X)},$$

where $N(X)$ is the number of elements in input set X.

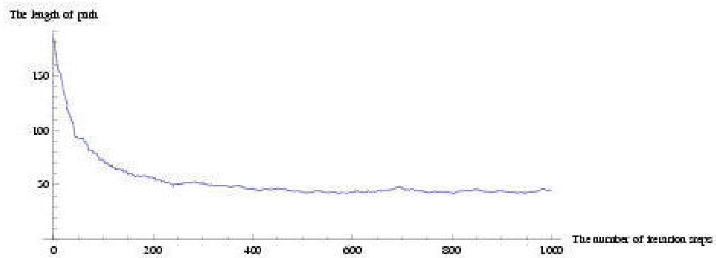


FIGURE 4. Transition of the length of path

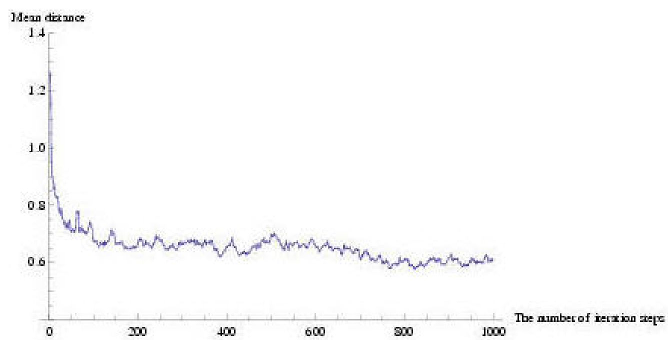


FIGURE 5. Transition of the mean distance MD

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