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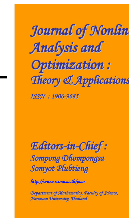

EXISTENCE, UNIQUENESS AND POSITIVITY OF SOLUTIONS FOR A NEUTRAL NONLINEAR PERIODIC DYNAMIC EQUATION ON A TIME SCALE

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ABSTRACT. Let $\mathbb{T}$ be a periodic time scale. We use Krasnoselskii's fixed point theorem, to show new results on the existence and positivity of solutions for the nonlinear periodic dynamic equation with variable delay of the form

$$x^{\triangle}(t) = -a(t)\,x(t) + \left(Q(t, x(g(t)))\right)^{\triangle} + G(t, x(t), x(g(t))),$$
$$x(t+T) = x(t).$$

Also, by transforming the problem to an integral equation we are able, using the contraction mapping principle, to show that the periodic solution is unique.

KEYWORDS : Fixed point theory, Nonlinear neutral dynamic equation, Periodic solutions, Positivity, Time scales.
AMS Subject Classification: 34K13, 34K30, 34L30.

1. INTRODUCTION

In recent years, there have been a few papers written on the existence of periodic solutions, nontrivial periodic solutions and positive periodic solutions for several classes of functional differential and dynamic equations with delays, which arise from a number of mathematical ecological models, economical and control models, physiological and population models and other models, see the references [1]-[13] and references therein.

Let $\mathbb{T}$ be a periodic time scale such that $0 \in \mathbb{T}$. In this paper, we are interested in the analysis of qualitative theory of periodic solutions of dynamic equations. Motivated by the papers [1, 2, 3, 4, 5, 6, 11, 12, 13] and the references therein, we consider the following nonlinear neutral periodic dynamic equation with variable delay

$$x^{\triangle}(t) = -a(t)\,x(t) + \left(Q(t, x(g(t)))\right)^{\triangle} + G(t, x(t), x(g(t))),$$
$$x(t+T) = x(t), \tag{1.1}$$

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where $T > 0$ be fixed, the nonlinear terms Q and G are $L^1_\triangle$-Carathéodory functions and the function $a \in L^1[0, T]$ is bounded. Throughout this paper we assume that $g : \mathbb{T} \to \mathbb{T}$ is strictly increasing so that the function $x(g(t))$ is well defined over $\mathbb{T}$. Our purpose here is to use the Krasnoselskii's fixed point theorem to show the existence and positivity of solutions on time scales for periodic dynamic equation (1.1). We have to transform (1.1) into an integral equation written as a sum of two mapping; one is a contraction and the other is a completely continuous operator. After that, we use the Krasnoselskii fixed point theorem, to show the existence and positivity of periodic solutions for equation (1.1). Also, transforming equation (1.1) to an integral equation enables us to show the uniqueness of the periodic solution by appealing to the contraction mapping principle.

A special case of equation (1.1) with $\mathbb{T} = \mathbb{R}$, in [2] we have investigated the existence, uniqueness and positivity of periodic solution for equation (1.1) by the Krasnoselskii's fixed point theorem and the contraction mapping principle. The results presented in this paper extend the main results in [2].

In Section 2, we present some preliminary material that we will need through the remainder of the paper. We will state some facts about the exponential function on a time scale as well as the Krasnoselskii's fixed point theorem. For details on Krasnoselskii's theorem we refer the reader to [14]. Also we present the inversion of neutral nonlinear periodic dynamic equation (1.1). In Section 3, we present our main results on existence, uniqueness and positivity.

2. PRELIMINARIES

A time scale is an arbitrary nonempty closed subset of real numbers. The study of dynamic equations on time scales is a fairly new subject, and research in this area is rapidly growing (see [1], [4], [6]-[10], [11], [12] and papers therein). The theory of dynamic equations unifies the theories of differential equations and difference equations. We suppose that the reader is familiar with the basic concepts concerning the calculus on time scales for dynamic equations. Otherwise one can find in Bohner and Peterson books [9] and [10] most of the material needed to read this paper. We start by giving some definitions necessary for our work. The notion of periodic time scales is introduced in Atici et al. [7] and Kaufmann and Raffoul [11]. The following two definitions are borrowed from [7] and [11].

Definition 2.1. We say that a time scale $\mathbb{T}$ is periodic if there exist a $\omega > 0$ such that if $t \in \mathbb{T}$ then $t \pm \omega \in \mathbb{T}$. For $\mathbb{T} \neq \mathbb{R}$, the smallest positive ω is called the period of the time scale.

Below are examples of periodic time scales taken from [11].

Example 2.2. The following time scales are periodic.

(1) $\mathbb{T} = \bigcup_{i=-\infty}^{\infty} [2(i-1)h, 2ih]$, $h > 0$ has period $\omega = 2h$.
(2) $\mathbb{T} = h\mathbb{Z}$ has period $\omega = h$.
(3) $\mathbb{T} = \mathbb{R}$.
(4) $\mathbb{T} = \{t = k - q^m : k \in \mathbb{Z}, m \in \mathbb{N}_0\}$ where, $0 < q < 1$ has period $\omega = 1$.

Remark 2.3 ([11])**.** All periodic time scales are unbounded above and below.

Definition 2.4. Let $\mathbb{T} \neq \mathbb{R}$ be a periodic time scales with the period ω. We say that the function $f : \mathbb{T} \to \mathbb{R}$ is periodic with period T if there exists a natural number n such that $T = n\omega$, $f(t \pm T) = f(t)$ for all $t \in \mathbb{T}$ and T is the smallest number such that $f(t \pm T) = f(t)$. If $\mathbb{T} = \mathbb{R}$, we say that f is periodic with period $T > 0$ if T is the smallest positive number such that $f(t \pm T) = f(t)$ for all $t \in \mathbb{T}$.

Remark 2.5 ([11]). If $\mathbb{T}$ is a periodic time scale with period p, then $\sigma(t \pm n\omega) = \sigma(t) \pm n\omega$. Consequently, the graininess function μ satisfies $\mu(t \pm n\omega) = \sigma(t \pm n\omega) - (t \pm n\omega) = \sigma(t) - t = \mu(t)$ and so, is a periodic function with period ω.

Our first two theorems concern the composition of two functions. The first theorem is the chain rule on time scales ([9], Theorem 1.93).

Theorem 2.6 (Chain Rule). *Assume $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. Let $\omega : \widetilde{\mathbb{T}} \to \mathbb{R}$. If $\nu^{\triangle}(t)$ and $\omega^{\widetilde{\triangle}}(\nu(t))$ exist for $t \in \mathbb{T}^k$, then*

$$(\omega \circ \nu)^{\triangle} = \left(\omega^{\widetilde{\triangle}} \circ \nu\right) \nu^{\triangle}.$$

In the sequel we will need to differentiate and integrate functions of the form $f(t - r(t)) = f(\nu(t))$ where, $\nu(t) := t - r(t)$. Our second theorem is the substitution rule ([9], Theorem 1.98).

Theorem 2.7 (Substitution). *Assume $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. If $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous function and ν is differentiable with rd-continuous derivative, then for $a, b \in \mathbb{T}$,*

$$\int_a^b f(t)\, \nu^{\triangle}(t) \triangle t = \int_{\nu(a)}^{\nu(b)} \left(f \circ \nu^{-1}\right)(s)\, \widetilde{\triangle} s.$$

A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive provided $1 + \mu(t)\, p(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive rd-continuous function $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$ while the set $\mathcal{R}^+ = \{f \in \mathcal{R} : 1 + \mu(t)\, f(t) > 0 \text{ for all } t \in \mathbb{T}\}$.

Let $p \in \mathcal{R}$ and $\mu(t) \neq 0$ for all $t \in \mathbb{T}$. The exponential function on $\mathbb{T}$ is defined by

$$e_p(t, s) = \exp\left(\int_s^t \frac{1}{\mu(z)} Log\left(1 + \mu(z)\, p(z)\right) \triangle z\right). \tag{2.1}$$

It is well known that if $p \in \mathcal{R}^+$, then $e_p(t, s) > 0$ for all $t \in \mathbb{T}$. Also, the exponential function $y(t) = e_p(t, s)$ is the solution to the initial value problem $y^{\triangle} = p(t)\, y$, $y(s) = 1$. Other properties of the exponential function are given in the following lemma.

Lemma 2.8 ([9]). *Let $p, q \in \mathcal{R}$. Then*
(i) $e_0(t, s) = 1$ *and* $e_p(t, t) = 1$;
(ii) $e_p(\sigma(t), s) = (1 + \mu(t)\, p(t))\, e_p(t, s)$;
(iii) $\dfrac{1}{e_p(t, s)} = e_{\ominus p}(t, s)$, *where* $\ominus p(t) = -\dfrac{p(t)}{1 + \mu(t)\, p(t)}$;
(iv) $e_p(t, s) = \dfrac{1}{e_p(s, t)} = e_{\ominus p}(s, t)$;
(v) $e_p(t, s)\, e_p(s, r) = e_p(t, r)$;
(vi) $e_p^{\triangle}(., s) = p e_p(., s)$ *and* $\left(\dfrac{1}{e_p(., s)}\right)^{\triangle} = -\dfrac{p(t)}{e_p^{\sigma}(., s)}$.

Theorem 2.9 ([8], Theorem 2.1). *Let $\mathbb{T}$ be a periodic time scale with period $\omega > 0$. If $p \in C_{rd}(\mathbb{T})$ is a periodic function with the period $T = n\omega$, then*

$$\int_{a+T}^{b+T} p(u) \triangle u = \int_a^b p(u) \triangle u, \;\; e_p(b + T, a + T) = e_p(b, a) \;\; \text{if } p \in \mathcal{R},$$

and $e_p(t + T, t)$ is independent of $t \in \mathbb{T}$ whenever $p \in \mathcal{R}$.

Lemma 2.10 ([1]). *If $p \in \mathcal{R}^+$, then*

$$0 < e_p(t,s) \le \exp\left(\int_s^t p(u) \triangle u\right), \ \forall t \in \mathbb{T}.$$

Corollary 2.11 ([1]). *If $p \in \mathcal{R}^+$ and $p(t) < 0$ for all $t \in \mathbb{T}$, then for all $s \in \mathbb{T}$ with $s \le t$ we have*

$$0 < e_p(t,s) \le \exp\left(\int_s^t p(u) \triangle u\right) < 1.$$

We state Krasnoselskii's fixed point theorem which enables us to prove the existence and positivity of periodic solutions to (1.1). For its proof we refer the reader to [14].

Theorem 2.12 (Krasnoselskii). *Let $\mathbb{M}$ be a closed convex nonempty subset of a Banach space $(S, \|\cdot\|)$. Suppose that A and B map $\mathbb{M}$ into S such that*
(i) $x, y \in \mathbb{M}$, *implies* $Ax + By \in \mathbb{M}$,
(ii) A *is completely continuous,*
(iii) B *is a contraction mapping.*
Then there exists $z \in \mathbb{M}$ with $z = Az + Bz$.

Let $T > 0$, $T \in \mathbb{T}$ be fixed and if $\mathbb{T} \ne \mathbb{R}$, $T = np$ for some $n \in \mathbb{N}$. By the notation $[a,b]$ we mean

$$[a,b] = \{t \in \mathbb{T} : a \le t \le b\},$$

unless otherwise specified. The intervals $[a,b)$, $(a,b]$ and (a,b) are defined similarly.

Define $P_T = \{\varphi : \mathbb{T} \to \mathbb{R} \mid \varphi \in C \text{ and } \varphi(t+T) = \varphi(t)\}$ where C is the space of continuous real-valued functions on $\mathbb{T}$. Then $(P_T, \|\cdot\|)$ is a Banach space with the supremum norm

$$\|\varphi\| = \sup_{t \in \mathbb{T}} |\varphi(t)| = \sup_{t \in [0,T]} |\varphi(t)|.$$

We will need the following lemma whose proof can be found in [11].

Lemma 2.13. *Let $x \in P_T$. Then $\|x^\sigma\| = \|x \circ \sigma\|$ exists and $\|x^\sigma\| = \|x\|$.*

The following definition is essential in our analysis.

Definition 2.14. A function $F : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ is an $L^1_\triangle$-Carathéodory function if it satisfies the following conditions

$(c1)$ For each $z \in \mathbb{R}^n$, the mapping $t \to F(t,z)$ is $\triangle$-measurable.
$(c2)$ For almost all $t \in [0,T]$, the mapping $t \to F(t,z)$ is continuous on $\mathbb{R}^n$.
$(c3)$ For each $r > 0$, there exists $f_r \in L^1_\triangle([0,T], \mathbb{R})$ such that for almost all $t \in [0,T]$ and for all z such that $|z| < r$, we have $|F(t,z)| \le f_r(t)$.

In this paper we use the notation $\gamma = -a$ and assume that $\gamma \in \mathcal{R}^+$ and will assume that the following conditions hold.

$(h1)$ $\gamma \in L^1_\triangle([0,T], \mathbb{R})$ is bounded, satisfies $\gamma(t+T) = \gamma(t)$ for all t and

$$1 - e_\gamma(t, t-T) \equiv \frac{1}{\eta} \ne 0.$$

$(h2)$ $g \in P_T$.
$(h3)$ Q and G are $L^1_\triangle$-Carathéodory functions, and $Q(t+T, x) = Q(t,x)$, $G(t+T, x, y) = Q(t, x, y)$ for all t.

We have to invert equation (1.1). For this, we use the variation of parameter formula to rewrite the equation as an integral equation suitable for Krasnoselskii theorem and the contraction mapping principle.

Lemma 2.15. *Suppose (h1)-(h3) hold. If $x \in P_T$, then x is a solution of equation (1.1) if and only if*

$$x(t) = Q(t, x(g(t))) + \eta \int_{t-T}^{t} [G(s, x(s), x(g(s))) + \gamma(s) Q(s, x(g(s)))] e_{\gamma}(t, \sigma(s)) \triangle s. \tag{2.2}$$

Proof. Let $x \in P_T$ be a solution of (1.1). First we write this equation as

$$(x(t) - Q(t, x(g(t))))^{\triangle} - \gamma(t)(x(t) - Q(t, x(g(t)))) = G(t, x(t), x(g(t))) + \gamma(t) Q(t, x(g(t))).$$

Multiply both sides of the above equation by $e_{\ominus\gamma}(\sigma(t), 0)$ we get

$$\left\{(x(t) - Q(t, x(g(t))))^{\triangle} - \gamma(t)(x(t) - Q(t, x(g(t))))\right\} e_{\ominus\gamma}(\sigma(t), 0) = \{G(t, x(t), x(g(t))) + \gamma(t) Q(t, x(g(t)))\} e_{\ominus\gamma}(\sigma(t), 0).$$

Since $e_{\ominus\gamma}(t, 0)^{\triangle} = -\gamma(t) e_{\ominus\gamma}(\sigma(t), 0)$ we find

$$[(x(t) - Q(t, x(g(t)))) e_{\ominus\gamma}(t, 0)]^{\triangle} = [G(t, x(t), x(g(t))) + \gamma(t) Q(t, x(g(t)))] e_{\ominus\gamma}(\sigma(t), 0).$$

Taking the integral from $t - T$ to t, we obtain

$$\int_{t-T}^{t} [(x(s) - Q(s, x(g(s)))) e_{\ominus\gamma}(s, 0)]^{\triangle} \triangle s = \int_{t-T}^{t} [G(s, x(s), x(g(s))) + \gamma(s) Q(s, x(g(s)))] e_{\ominus\gamma}(\sigma(s), 0) \triangle s.$$

As a consequence, we arrive at

$$(x(t) - Q(t, x(g(t)))) e_{\ominus\gamma}(t, 0) - (x(t-T) - Q(t-T, x(g(t-T)))) e_{\ominus\gamma}(t-T, 0) = \int_{t-T}^{t} [G(s, x(s), x(g(s))) + \gamma(s) Q(s, x(g(s)))] e_{\ominus\gamma}(\sigma(s), 0) \triangle s.$$

Dividing both sides of the above equation by $e_{\ominus\gamma}(t, 0)$ and using the fact that $x(t-T) = x(t)$, (h1)-(h3) and

$$\frac{e_{\ominus\gamma}(t-T, 0)}{e_{\ominus\gamma}(t, 0)} = e_{\gamma}(t, t-T), \quad \frac{e_{\ominus\gamma}(\sigma(s), 0)}{e_{\ominus\gamma}(t, 0)} = e_{\gamma}(t, \sigma(s)),$$

we obtain

$$x(t) - Q(t, x(g(t))) = \eta \int_{t-T}^{t} [G(s, x(s), x(g(s))) + \gamma(s) Q(s, x(g(s)))] e_{\gamma}(t, \sigma(s)) \triangle s.$$

Since each step is reversible, the converse follows easily. This completes the proof. □

3. EXISTENCE RESULTS

We present our existence results in this section. To this end, we first define the operator H by

$$(H\varphi)(t) = Q(t,\varphi(g(t))) + \eta\int_{t-T}^{t}\left[G(s,\varphi(s),\varphi(g(s))) + \gamma(s)Q(s,\varphi(g(s)))\right]e_\gamma(t,\sigma(s))\triangle s, \tag{3.1}$$

From Lemma 2.15 we see that fixed points of H are solutions of (1.1) and vice versa. In order to employ Theorem 2.12 we need to express the operator H as the sum of two operators, one of which is completely continuous and the other of which is a contraction. Let $(H\varphi)(t) = (A\varphi)(t) + (B\varphi)(t)$ where

$$(A\varphi)(t) = \eta\int_{t-T}^{t}\left[G(s,\varphi(s),\varphi(g(s))) + \gamma(s)Q(s,\varphi(g(s)))\right]e_\gamma(t,\sigma(s))\triangle s, \tag{3.2}$$

and

$$(B\varphi)(t) = Q(t,\varphi(g(t))). \tag{3.3}$$

Our first lemma in this section shows that $A : P_T \to P_T$ is completely continuous.

Lemma 3.1. *Suppose that conditions* $(h1)-(h3)$ *hold. Then* $A : P_T \to P_T$ *is completely continuous.*

Proof. We first show that $A : P_T \to P_T$. Clearly, if φ is continuous, then $A\varphi$ is. From (3.2) and conditions $(h1)-(h3)$, it follows trivially that $e_\gamma(t+T,\sigma(s)+T) = e_\gamma(t,\sigma(s))$ by Theorem 2.9. Consequently, we have that

$$(A\varphi)(t+T) = (A\varphi)(t),$$

by Theorem 2.7. That is, if $\varphi \in P_T$ then $A\varphi \in P_T$.

To see that A is continuous. Let $\{\varphi_i\} \subset P_T$ be such that $\varphi_i \to \varphi$ as $i \to \infty$. By the Dominated Convergence Theorem,

$$\begin{aligned}
&\lim_{i\to\infty}|(A\varphi_i)(t) - (A\varphi)(t)| \\
&\le \lim_{i\to\infty}\eta\int_{t-T}^{t}\left[|G(s,\varphi_i(s),\varphi_i(g(s))) - G(s,\varphi(s),\varphi(g(s)))|\right. \\
&\left.+|\gamma(s)|\,|Q(s,\varphi_i(g(s))) - Q(s,\varphi(g(s)))|\right]e_\gamma(t,\sigma(s))\triangle s \\
&= \eta\int_{t-T}^{t}\Big[\lim_{i\to\infty}|G(s,\varphi_i(s),\varphi_i(g(s))) - G(s,\varphi(s),\varphi(g(s)))| \\
&+|\gamma(s)|\lim_{i\to\infty}|Q(s,\varphi_i(g(s))) - Q(s,\varphi(g(s)))|\Big]e_\gamma(t,\sigma(s))\triangle s \\
&= 0.
\end{aligned}$$

Hence $A : P_T \to P_T$ is continuous.

Finally, we show that A is completely continuous. Let $\mathfrak{B} \subset P_T$ be a closed bounded subset and let C be such that $\|\varphi\| \le C$ for all $\varphi \in \mathfrak{B}$. then

$$\begin{aligned}
|(A\varphi)(t)| &\le \eta\int_{t-T}^{t}\left[|G(s,\varphi(s),\varphi(g(s)))| + |\gamma(s)|\,|Q(s,\varphi(g(s)))|\right]e_\gamma(t,\sigma(s))\triangle s \\
&\le \eta N\left\{\int_{t-T}^{t}g_C(s)\triangle s + \int_{t-T}^{t}|\gamma(s)|\,q_C(s)\triangle s\right\}
\end{aligned}$$

$$\leq \eta N \left\{ \int_{t-T}^{t} g_C(s) \triangle s + \alpha \int_{t-T}^{t} q_C(s) \triangle s \right\} \equiv K,$$

where $N = \sup_{s\in[t-T,t]} e_\gamma(t,\sigma(s))$ and $\alpha = \sup_{s\in[t-T,t]} |\gamma(s)|$. And so, the family of functions $A\varphi$ is uniformly bounded.

Again, let $\varphi \in \mathfrak{B}$. Without loss of generality, we can pick $t_1 < t_2$ such that $t_2 - t_1 < T$. Then

$$\begin{aligned}
&|(A\varphi)(t_2) - (A\varphi)(t_1)| \\
&= \eta \left| \int_{t_2-T}^{t_2} [G(s,\varphi(s),\varphi(g(s))) + \gamma(s) Q(s,\varphi(g(s)))] e_\gamma(t,\sigma(s)) \triangle s \right. \\
&\left. - \int_{t_1-T}^{t_1} [G(s,\varphi(s),\varphi(g(s))) + \gamma(s) Q(s,\varphi(g(s)))] e_\gamma(t,\sigma(s)) \triangle s \right|.
\end{aligned}$$

We can rewrite the left hand side as the sum of three integrals. We obtain the following

$$\begin{aligned}
&|(A\varphi)(t_2) - (A\varphi)(t_1)| \\
&\leq \eta \int_{t_1}^{t_2} [|G(s,\varphi(s),\varphi(g(s)))| + |\gamma(s)| |Q(s,\varphi(g(s)))|] e_\gamma(t_2,\sigma(s)) \triangle s \\
&+ \eta \int_{t_2-T}^{t_1} [|G(s,\varphi(s),\varphi(g(s)))| + |\gamma(s)| |Q(s,\varphi(g(s)))|] \\
&\times |e_\gamma(t_2,\sigma(s)) - e_\gamma(t_1,\sigma(s))| \triangle s \\
&+ \eta \int_{t_1-T}^{t_2-T} [|G(s,\varphi(s),\varphi(g(s)))| + |\gamma(s)| |Q(s,\varphi(g(s)))|] e_\gamma(t_1,\sigma(s)) \triangle s \\
&\leq 2\eta N \left\{ \int_{t_1}^{t_2} g_C(s) + \alpha q_C(s) \triangle s \right\} \\
&+ \eta \int_{t_2-T}^{t_1} [g_C(s) + \alpha q_C(s)] |e_\gamma(t_2,\sigma(s)) - e_\gamma(t_1,\sigma(s))| \triangle s.
\end{aligned}$$

Now $\int_{t_1}^{t_2} g_C(s) + \alpha q_C(s) \triangle s \to 0$ as $(t_2 - t_1) \to 0$. Also, since

$$\begin{aligned}
&\int_{t_2-T}^{t_1} [g_C(s) + \alpha q_C(s)] |e_\gamma(t_2,\sigma(s)) - e_\gamma(t_1,\sigma(s))| \triangle s \\
&\leq \int_0^T [g_C(s) + \alpha q_C(s)] |e_\gamma(t_2,\sigma(s)) - e_\gamma(t_1,\sigma(s))| \triangle s,
\end{aligned}$$

and $|e_\gamma(t_2,\sigma(s)) - e_\gamma(t_1,\sigma(s))| \to 0$ as $(t_2 - t_1) \to 0$, then by the Dominated Convergence Theorem,

$$\int_{t_2-T}^{t_1} [g_C(s) + \alpha q_C(s)] |e_\gamma(t_2,\sigma(s)) - e_\gamma(t_1,\sigma(s))| \triangle s \to 0,$$

as $(t_2 - t_1) \to 0$. Thus $|(A\varphi)(t_2) - (A\varphi)(t_1)| \to 0$ as $(t_2 - t_1) \to 0$ independently of $\varphi \in \mathfrak{B}$. As such, the family of functions $A\varphi$ is equicontinuous on $\mathfrak{B}$.

By the Arzelà-Ascoli Theorem, A is completely continuous and the proof is complete. □

We need the following condition on the nonlinear term Q.

$(h4)$ the function $Q(t,x)$ is Lipschitz continuous in x. That is, there exists $L>0$ such that

$$|Q(t,x)-Q(t,y)| \leq L\|x-y\|.$$

Our next lemma gives a sufficient condition under which $B: P_T \to P_T$ is a contraction.

Lemma 3.2. *Suppose that conditions $(h3)$ and $(h4)$ hold and $L<1$. Then $B: P_T \to P_T$ is a contraction.*

Proof. From (3.3) and conditions $(h2)$ and $(h3)$, we have that

$$(B\varphi)(t+T) = (B\varphi)(t).$$

That is, if $\varphi \in P_T$ then $B\varphi \in P_T$.

To see that B is a contraction. Let $\varphi, \psi \in P_T$ we have

$$\begin{aligned}\|B(\varphi)-B(\psi)\| &= \sup_{t\in[0,T]} |(B\varphi)(t)-(B\psi)(t)| \\ &\leq L \sup_{t\in[0,T]} |\varphi(g(t))-\psi(g(t))| \\ &\leq L\|\varphi-\psi\|.\end{aligned}$$

Hence $B: P_T \to P_T$ is a contraction. □

We need the following conditions on the nonlinear terms Q and G.

$(h5)$ There exists periodic functions $q_1, q_2 \in L^1_\triangle[0,T]$, with period T, such that

$$|Q(t,x)| \leq q_1(t)|x| + q_2(t),$$

for all $x \in \mathbb{R}$.

$(h6)$ There exists periodic functions $g_1, g_2, g_3 \in L^1_\triangle[0,T]$, with period T, such that

$$|G(t,x,y)| \leq g_1(t)|x| + g_2(t)|y| + g_3(t),$$

for all $x, y \in \mathbb{R}$.

Also, we now define some quantities that will be used in the following theorem. Let $\delta = \sup_{t\in[0,T]} e_\gamma(t,\sigma(s))$, $\theta = \sup_{t\in[0,T]} |Q(t,0)|$, $\lambda = \int_0^T |q_1(s)|\triangle s$, $\mu = \int_0^T |q_2(s)|\triangle s$, $\beta = \int_0^T |g_1(s)|\triangle s$, $\gamma = \int_0^T |g_2(s)|\triangle s$, $\Gamma = \int_0^T |g_3(s)|\triangle s$.

Theorem 3.3. *Suppose that conditions $(h1)-(h6)$ hold and $L<1$. Suppose there exists a positive constant J satisfying the inequality*

$$\theta + \eta\delta(\alpha\mu+\Gamma) + (L+\eta\delta(\alpha\lambda+\beta+\gamma))J \leq J.$$

Then (1.1) has a solution $\varphi \in P_T$ such that $\|\varphi\| \leq J$.

Proof. Define $\mathbb{M} = \{\varphi \in P_T : \|\varphi\| \leq J\}$. By Lemma 3.1, the operator $A: \mathbb{M} \to P_T$ is completely continuous. Since $L<1$, then by Lemma 3.2, the operator $B: \mathbb{M} \to P_T$ is a contraction. Conditions (ii) and (iii) of Theorem 2.12 are satisfied. We need to show that condition (i) is fulfilled. To this end, let $\varphi, \psi \in \mathbb{M}$. Then

$$\begin{aligned}&|(A\varphi)(t)+(B\psi)(t)| \\ &\leq |Q(t,\psi(g(t)))| + \eta\int_{t-T}^t |G(s,\varphi(s),\varphi(g(s)))| e_\gamma(t,\sigma(s))\triangle s \\ &+ \eta\int_{t-T}^t |\gamma(s)||Q(s,\varphi(g(s)))| e_\gamma(t,\sigma(s))\triangle s \\ &\leq LJ + \theta + \eta\delta(\beta J+\gamma J+\Gamma) + \eta\alpha\delta(\lambda J+\mu)\end{aligned}$$

$$= \theta + \eta\delta\left(\alpha\mu + \Gamma\right) + \left(L + \eta\delta\left(\alpha\lambda + \beta + \gamma\right)\right) J \leq J.$$

Thus $\|A\varphi + B\psi\| \leq J$ and so $A\varphi + B\psi \in \mathbb{M}$. All the conditions of Theorem 2.12 are satisfied and consequently the operator H defined in (3.1) has a fixed point in $\mathbb{M}$. By Lemma 2.15 this fixed point is a solution of (1.1) and the proof is complete. □

The conditions $(h5)$ and $(h6)$ are global conditions on the functions Q and G. In the next theorem we replace this conditions with the following local conditions.

$(h5^*)$ There exists periodic functions $q_1^*, q_2^* \in L^1_\triangle [0,T]$, with period T, such that

$$|Q(t,x)| \leq q_1^*(t)|x| + q_2^*(t),$$

for all x with $|x| \leq J$.

$(h6^*)$ There exists periodic functions $g_1^*, g_2^*, g_3^* \in L^1_\triangle [0,T]$, with period T, such that

$$|G(t,x,y)| \leq g_1^*(t)|x| + g_2^*(t)|y| + g_3^*(t),$$

for all x, y with $|x| \leq J$ and $|y| \leq J$.
The constants λ^*, μ^* and β^*, γ^*, Γ^* are defined as before with the understanding that the functions q_1^*, q_2^* and g_1^*, g_2^*, g_3^* are those from conditions $(h5^*)$ and $(h6^*)$, respectively.

Theorem 3.4. *Suppose that conditions* $(h1) - (h4)$, $(h5^*)$ *and* $(h6^*)$ *hold and* $L < 1$. *Suppose there exists a positive constant* J *satisfying the inequality*

$$\theta + \eta\delta\left(\alpha\mu^* + \Gamma^*\right) + \left(L + \eta\delta\left(\alpha\lambda^* + \beta^* + \gamma^*\right)\right) J \leq J.$$

Then (1.1) has a solution $\varphi \in P_T$ *such that* $\|\varphi\| \leq J$.

The proof of the above theorem parallels that of Theorem 3.3.

For our next result, we give conditions for which there exists a unique solution of (1.1). We replace conditions $(h5)$ and $(h6)$ with the following conditions.

$(h5^\dagger)$ There exists periodic function $q_1^\dagger \in L^1_\triangle [0,T]$, with period T, such that

$$|Q(t,x) - Q(t,x)| \leq q_1^\dagger(t)|x - y|,$$

for all $x, y \in \mathbb{R}$.

$(h6^\dagger)$ There exists periodic functions $g_1^\dagger, g_2^\dagger \in L^1_\triangle [0,T]$, with period T, such that

$$|G(t,x,y) - G(t,z,w)| \leq g_1^\dagger(t)|x - z| + g_2^\dagger(t)|y - w|,$$

for all $x, y, z, w \in \mathbb{R}$.
The constants $\lambda^\dagger$ and $\beta^\dagger$, $\gamma^\dagger$ are defined as before with the understanding that the functions $q_1^\dagger$ and $g_1^\dagger$, $g_2^\dagger$ are those from conditions $(h5^\dagger)$ and $(h6^\dagger)$, respectively.

Theorem 3.5. *Suppose that conditions* $(h1) - (h4)$, $(h5^\dagger)$ *and* $(h6^\dagger)$ *hold. If*

$$L + \eta\delta\left(\alpha\lambda^\dagger + \beta^\dagger + \gamma^\dagger\right) < 1,$$

then (1.1) has a unique T*-periodic solution.*

Proof. Let $\varphi, \psi \in P_T$. By (3.1) we have for all t,

$$\begin{aligned}
&|(H\varphi)(t) - (H\psi)(t)| \\
&\leq |Q(t,\varphi(g(t))) - Q(t,\psi(g(t)))| \\
&+ \eta \int_{t-T}^{t} [|G(s,\varphi(s),\varphi(g(s))) - G(s,\psi(s),\psi(g(s)))| \\
&+ |\gamma(s)|\,|Q(s,\varphi(g(s))) - Q(s,\psi(g(s)))|]\, e_\gamma(t,\sigma(s)) \triangle s \\
&\leq L\|\varphi - \psi\| + \eta\delta\left(\alpha\lambda^\dagger + \beta^\dagger + \gamma^\dagger\right)\|\varphi - \psi\|.
\end{aligned}$$

Hence, $\|H\varphi - H\psi\| \leq \left(L + \eta\delta\left(\alpha\lambda^{\dagger} + \beta^{\dagger} + \gamma^{\dagger}\right)\right)\|\varphi - \psi\|$. By the contraction mapping principle, H has a fixed point in P_T and by Lemma 2.15, this fixed point is a solution of (1.1). The proof is complete. □

For our last result, we give sufficient conditions under which there exists positive solutions of equation (1.1). We begin by defining some new quantities. Let

$$m = \min_{s\in[t-T,t]} e_{\gamma}(t, \sigma(s)), \ M = \max_{s\in[t-T,t]} e_{\gamma}(t, \sigma(s)).$$

Given constants $0 < \mathfrak{L} < \mathfrak{K}$, define the set $\mathbb{M}_1 = \{\varphi \in P_T : \mathfrak{L} \leq \varphi(t) \leq \mathfrak{K}, \ t \in [0, T]\}$.

Assume the following conditions hold.

$(h7)$ There exists constants $0 < q^* < L^*$ such that $q^*\mathfrak{L} \leq Q(t, \rho) \leq L^*\mathfrak{K}$ for all $\rho \in \mathbb{M}_1$ and $t \in [0, T]$.

$(h8)$ There exists constants $0 < \mathfrak{L} < \mathfrak{K}$ such that

$$\frac{(1-q^*)\mathfrak{L}}{\eta m T} \leq G(s, \sigma, \rho) + \gamma(s) Q(s, \rho) \leq \frac{(1-L^*)\mathfrak{K}}{\eta M T},$$

for all $\sigma, \rho \in \mathbb{M}_1$ and $s \in [t-T, t]$.

Theorem 3.6. *Suppose that conditions $(h1) - (h4)$, $(h7)$ and $(h8)$ hold and $L < 1$. Then there exists a positive periodic solution of (1.1).*

Proof. As in the proof of Theorem 3.3, we just need to show that condition (i) of Theorem 2.12 is satisfied. Let $\varphi, \psi \in \mathbb{M}_1$. Then

$$\begin{aligned}
&(A\varphi)(t) + (B\psi)(t) \\
&= Q(t, \psi(g(t))) + \eta \int_{t-T}^{t} [G(s, \varphi(s), \varphi(g(s))) + \gamma(s) Q(s, \varphi(g(s)))] e_{\gamma}(t, \sigma(s)) \triangle s \\
&\geq q^*\mathfrak{L} + \eta m T \frac{(1-q^*)\mathfrak{L}}{\eta m T} = \mathfrak{L}.
\end{aligned}$$

Likewise

$$(A\varphi)(t) + (B\psi)(t) \leq L^*\mathfrak{K} + \eta M T \frac{(1-L^*)\mathfrak{K}}{\eta M T} = \mathfrak{K}.$$

By Theorem 2.12, the operator H has a fixed point in $\mathbb{M}_1$. This fixed point is a positive solution of (1.1) and the proof is complete. □

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