
**BALL CONVERGENCE RESULTS FOR A METHOD WITH MEMORY OF
EFFICIENCY INDEX 1.8392 USING ONLY FUNCTIONAL VALUES**

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ABSTRACT. We present a local convergence analysis for an one-step iterative method with memory of efficiency index 1.8392 to solve nonlinear equations. If the function is twice differentiable, then it was shown that the R -order of convergence is 1.8392. In this paper we use hypotheses up to the first derivative. This way we extend the applicability of this method. Moreover, the radius of convergence and computable error bounds on the distances involved are also given in this study. Numerical examples are also presented to illustrate the theoretical results.

KEYWORDS : Halley's method; local convergence; order of convergence; efficiency index.

AMS Subject Classification: 65D10, 65D99

1. INTRODUCTION

Let $f : D \subseteq S \rightarrow S$ be a nonlinear function, D is a convex subset of S where S is \mathbb{R} or \mathbb{C} . Consider the problem of approximating a locally unique solution x^* of equation

$$f(x) = 0. \quad (1.1)$$

Newton-like methods are famous for finding solution of (1.1), these methods are usually studied based on: semi-local and local convergence. The semi-local convergence matter is, based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls [3, 4, 19, 20, 23].

Third order methods such as Euler's, Halley's, super Halley's, Chebyshev's [1]-[23] require the evaluation of the second derivative f'' at each step, which in general is very expensive. That is why many authors have used higher order multipoint

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methods [1]-[23]. In this paper, we study the local convergence of the one-step method with memory defined for each $n = 0, 1, 2, \dots$ by

$$x_{n+1} = \frac{(x_n + x_{n-1})f_n f_{n-1} f[x_n, x_{n-1}]}{2f_n f_{n-1} f[x_n, x_{n-1}] - (f_{n-1}^2 f[x_n, x_{n-1}] + f_n^2 f[x_{n-1}, x_{n-2}])}, \\ - \frac{(f_{n-1}^2 x_n f[x_n, x_{n-1}] + f_n^2 x_{n-1} f[x_{n-1}, x_{n-2}])}{2f_n f_{n-1} f[x_n, x_{n-1}] - (f_{n-1}^2 f[x_n, x_{n-1}] + f_n^2 f[x_{n-1}, x_{n-2}])}, \quad (1.2)$$

where x_{-2}, x_{-1}, x_0 are initial points, $f_n = f(x_n)$, $f[x, y]$ denotes a divided difference of order one for function f at the point x, y [3, 4, 23] defined by

$$f[x, y] = \frac{f(x) - f(y)}{x - y} \quad \text{if } x \neq y \quad (1.3)$$

and $f[x, x] = f'(x)$. Method (1.2) uses only one function evaluation per step, f_n and it was shown using the Herberger's matrix method in [17] that the R -order of convergence is $\frac{1}{3}(1 + \sqrt[3]{19 - 3\sqrt{33}} + \sqrt[3]{19 + 3\sqrt{33}}) \approx 1.8392$. Moreover, the efficiency index is

$$I = (1.8392)^{\frac{1}{3}} = 1.8392.$$

These results are obtained provided that the function f is twice differentiable. This hypotheses limits the applicability of method (1.2) although only function evaluations are needed to carry out the computation of each step. As a motivational example, let us define function f on $D = [-1, 2]$ by

$$f(x) = \begin{cases} x^2 \ln x + x^4 - x^3, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Then, we have that

$$f'(x) = 2x \ln x + 4x^3 - 3x^2 + x,$$

and

$$f''(x) = 2 \ln x + 12x^2 - 6x + 3.$$

Hence, function f'' is unbounded on D . In the present paper, we study the local convergence of method (1.2) using hypotheses only on the first derivative. Moreover, we provide: the radius of convergence, computable error bounds on the distances $|x_n - x^*|$ and a uniqueness result.

The rest of the paper is organized as follows: Section 2 contains the local convergence analysis of methods (1.2). The numerical examples are given in the concluding Section 3.

2. LOCAL CONVERGENCE FOR METHOD (1.2)

We present the local convergence analysis of method (1.2) in this section. We first simplify method (1.2) by using formula (1.3) to obtain that

$$x_{n+1} - x^* = \frac{A_n}{B_n}, \quad (2.1)$$

where

$$A_n = f_{n-1}(f_n - f_{n-1})^2(x_{n-1} - x_{n-2})(x_n - x^*) \\ + f_n[-(f_n - f_{n-1})^2(x_{n-1} - x_{n-2}) \\ + f_n((f_n - f_{n-1})(x_{n-1} - x_{n-2}) \\ - (f_{n-1} - f_{n-2})(x_n - x_{n-1}))(x_{n-1} - x^*)] \quad (2.2)$$

and

$$B_n = 2f_n f_{n-1} (f_n - f_{n-1})(x_{n-1} - x_{n-2}) - f_{n-1}^2 (f_n - f_{n-1})(x_{n-1} - x_{n-2}) - f_n^2 (f_n - f_{n-1})(x_n - x_{n-1}). \quad (2.3)$$

Let us also define function g by

$$g(t) = 40M^3 t^4 + 4M^3 t^2 - 1 \text{ for some } M > 0. \quad (2.4)$$

Notice that

$$r = \frac{1}{2M} \sqrt{\frac{2M}{M^2 + \sqrt{M^4 + 10M}}} \quad (2.5)$$

is the only positive root of equation $g(t) = 0$. We also have that

$$0 \leq \frac{24M^3 t^4}{1 - 4M^3 t^2(1 + 4t^2)} < 1 \text{ for each } t \in [0, r). \quad (2.6)$$

Let $U(v, \rho), \bar{U}(v, \rho)$ stand for the open and closed balls in S , respectively, with center $v \in S$ and of radius $\rho > 0$.

Using the preceding notation we present the local convergence analysis of method (1.2).

THEOREM 2.1. *Let $f : D \subseteq S \rightarrow S$ be a differentiable function. Suppose that there exist $x^* \in D, M > 0$ such that for each $x \in D$*

$$f(x^*) = 0, \quad (2.7)$$

$$|f'(x)| \leq M, \quad (2.8)$$

and

$$\bar{U}(x^*, r) \subseteq D, \quad (2.9)$$

where r is defined by (2.5). Then, the sequence $\{x_n\}$ generated by method (1.2) for $x_{-2}, x_{-1}, x_0 \in U(x^*, r)$ is well defined, remains in $U(x^*, r)$ for each $n = 0, 1, 2, \dots$ and converges to x^* . Moreover, the following estimates hold for each $n = 0, 1, 2, \dots$,

$$|x_{n+1} - x^*| \leq \frac{a_n}{b_n} |x_{n-1} - x^*| |x_n - x^*| < |x_n - x^*| < r, \quad (2.10)$$

where

$$a_n = 2M^3 (|x_n - x^*| + |x_{n-1} - x^*|) (|x_{n-1} - x^*| + |x_{n-2} - x^*|) (2|x_n - x^*| + |x_{n-1} - x^*|)$$

and

$$b_n = 1 - M^3 (|x_n - x^*| + |x_{n-1} - x^*|) (|x_{n-1} - x^*| + |x_{n-2} - x^*|) (1 + (|x_{n-1} - x^*| + |x_n - x^*|)^2).$$

Proof. By hypothesis $x_{-2}, x_{-1}, x_0 \in U(x^*, r)$. We can write using (2.7) that

$$f_{-2} = f_{-2} - f(x^*) = \int_0^1 f'(x^* + \theta(x_{-2} - x^*)) (x_{-2} - x^*) d\theta. \quad (2.11)$$

Then, by (2.11) and (2.8) we get that

$$|f_{-2}| = \left| \int_0^1 f'(x^* + \theta(x_{-2} - x^*)) (x_{-2} - x^*) d\theta \right| \leq M |x_{-2} - x^*|. \quad (2.12)$$

Similarly, we have that

$$|f_{-1}| \leq M |x_{-1} - x^*|. \quad (2.13)$$

and

$$|f_0| \leq M |x_0 - x^*|. \quad (2.14)$$

We shall show that B_0 given by (2.3) is invertible. We have in turn by (2.3), (2.6), (2.12)- (2.14) and the triangle inequality that

$$\begin{aligned}
|B_0 - 1| &= |2f_0f_{-1}(f_0 - f_{-1})(x_{-1} - x_{-2}) - f_{-1}^2(f_0 - f_{-1})(x_{-1} - x_{-2}) \\
&\quad - f_0^2(f_{-1} - f_{-2})(x_0 - x_{-1}) - (x_0 - x_{-1})(x_{-1} - x_{-2})| \\
&= |f_{-1}(f_0 - f_{-1})^2(x_{-1} - x_{-2}) + f_0(f_{-1}(f_0 - f_{-1})(x_{-1} - x_{-2}) \\
&\quad - f_0(f_{-1} - f_{-2})(x_0 - x_{-1})) - (x_0 - x_{-1})(x_{-1} - x_{-2})| \\
&\leq M|x_{-1} - x^*|M^2(|x_0 - x^*| + |x_{-1} - x^*|)^2(|x_{-1} - x^*| + |x_{-2} - x^*|) \\
&\quad + M|x_0 - x^*|(M|x_{-1} - x^*|M(|x_0 - x^*| + |x_{-1} - x^*|) \\
&\quad \times (|x_{-1} - x^*| + |x_{-2} - x^*|) \\
&\quad + M|x_0 - x^*|M(|x_{-1} - x^*| + |x_{-2} - x^*|)(|x_0 - x^*| + |x_{-1} - x^*|)) \\
&\quad + (|x_0 - x^*| + |x_{-1} - x^*|)(|x_{-1} - x^*| + |x_{-2} - x^*|) \\
&= 1 - b_0 < 4M^3r^2(1 + 4r^2) < 1. \tag{2.15}
\end{aligned}$$

It follows from (2.15) and the Banach lemma on invertible functions [3, 4] that B_0 is invertible and

$$|B_0^{-1}| \leq \frac{1}{b_0} < \frac{1}{1 - 4M^3r^2(1 + 4r^2)}. \tag{2.16}$$

Next, we need an estimate on A_0 . It follows from (2.2), (2.12)-(2.14) and triangle inequality that

$$\begin{aligned}
|A_0| &\leq M|x_{-1} - x^*|M^2(|x_{-1} - x^*| + |x_0 - x^*|)^2 \\
&\quad \times (|x_{-1} - x^*| + |x_{-2} - x^*|)|x_0 - x^*| \\
&\quad + M|x_0 - x^*||x_{-1} - x^*|[M^2(|x_0 - x^*| + |x_{-1} - x^*|)^2 \\
&\quad \times (|x_{-1} - x^*| + |x_{-2} - x^*|) \\
&\quad + M|x_0 - x^*|(M(|x_0 - x^*| + |x_{-1} - x^*|)(|x_{-1} - x^*| + |x_{-2} - x^*|) \\
&\quad + M(|x_{-1} - x^*| + |x_{-2} - x^*|)(|x_0 - x^*| + |x_{-1} - x^*|)] \\
&\leq a_0|x_{-1} - x^*||x_0 - x^*| \leq 24M^3r^4|x_0 - x^*|. \tag{2.17}
\end{aligned}$$

Then, using (2.1), (2.6), (2.16) and (2.17), we get that

$$\begin{aligned}
|x_{n+1} - x^*| &\leq \frac{a_0|x_{-1} - x^*||x_0 - x^*|}{b_0} \\
&< \frac{24M^3r^4}{1 - 4M^3r^2(1 + 4r^2)}|x_0 - x^*| = |x_0 - x^*| < r,
\end{aligned}$$

which shows (2.10) for $n = 0$ and $x_1 \in U(x^*, r)$. By simply replacing x_{-2}, x_{-1}, x_0 by x_{k-2}, x_{k-1}, x_k in the preceding estimates, we arrive at (2.10). Then from the estimate $|x_{k+1} - x^*| < |x_k - x^*| < r$, we deduce that $\lim_{k \rightarrow \infty} x_k = x^*$ and $x_{k+1} \in U(x^*, r)$.

Next, we present a uniqueness result for method (1.2).

THEOREM 2.2. *Suppose that the hypotheses of Theorem 2.1 hold and there exists $L_0 > 0$ such that for each $x \in D$, $f'(x^*) \neq 0$,*

$$|f'(x^*)^{-1}(f'(x) - f'(x^*))| \leq L_0|x - x^*| \tag{2.18}$$

and

$$L_0r < 2. \tag{2.19}$$

Then, the limit point x^* is the only solution of equation $f(x) = 0$ in $\bar{U}(x^*, r)$.

Proof. The existence of the solution x^* in $U(x^*, r)$ has been established in Theorem 2.1. To show the uniqueness part, let $y^* \in \bar{U}(x^*, T)$ with $f(y^*) = 0$. Define $T = \int_0^1 f'(y^* + \theta(x^* - y^*))d\theta$. Using (2.18) and (2.19) we get that

$$\begin{aligned} |f'(x^*)^{-1}(T - f'(x^*))| &\leq \int_0^1 L_0|y^* + \theta(x^* - y^*) - x^*|d\theta \\ &\leq \int_0^1 (1 - \theta)|x^* - y^*|d\theta \leq \frac{L_0}{2}r < 1. \end{aligned} \quad (2.20)$$

It follows from (2.20) that T is invertible. Finally, from the identity $0 = f(x^*) - f(y^*) = T(x^* - y^*)$, we deduce that $x^* = y^*$. \square

REMARK 2.3. The computation of the order of convergence involves estimates of higher order derivative of operator f . So one may use the computational order of convergence (COC) defined by

$$\xi = \ln \left(\frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|} \right) / \ln \left(\frac{\|x_n - x^*\|}{\|x_{n-1} - x^*\|} \right)$$

or the approximate computational order of convergence

$$\xi_1 = \ln \left(\frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|} \right) / \ln \left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|} \right).$$

3. NUMERICAL EXAMPLES

We present a numerical example in this section.

EXAMPLE 3.1. Let $D = [-1, 1]$. Define function f of D by

$$f(x) = e^x - 1. \quad (3.1)$$

Using (3.1) and $x^* = 0$, we get that $M = e$, $L_0 = e - 1$, $r = 0.1058$ and $\xi_1 = 1.5586$. Notice that (2.19) is satisfied, so the solution $x^* = 0$ is unique in $\bar{U}(0, 1)$.

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