DUALITY FOR A NONDIFFERENTIABLE MULTIOBJECTIVE HIGHER-ORDER SYMMETRIC FRACTIONAL PROGRAMMING PROBLEMS WITH CONE CONSTRAINTS

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ABSTRACT. In this paper, a pair of nondifferentiable multiobjective higher-order symmetric fractional dual problem with cone constraints is formulated. For a differentiable function, we introduce the definition of higher-order \((C, \alpha, \rho, d)\)-convexity. Next, we prove appropriate duality relations under aforesaid assumptions.

KEYWORDS: Higher-order; symmetric duality; multiobjective fractional programming; \((C, \alpha, \rho, d)\)-convexity; cone constraints.

AMS Subject Classification: 90C26 . 90C30 . 90C32 . 90C46

1. INTRODUCTION

Higher-order duality is significant due to its computational importance as it provides more higher bounds whenever approximation is used. By introducing two different functions, \(h : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}\) and \(k : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^m\), Mangasarian [8] formulated higher-order dual for a single objective nonlinear problems, \(\{\min f(x), \text{ subject to } g(x) \leq 0\}\). Inspired by this concept, many researchers have worked in this direction. Chen [1] has formulated higher-order multiobjective symmetric dual programs and established duality relations under higher-order \(F\)-convexity assumptions. A higher-order vector optimization problem and its dual has been studied by Kassem [9].

In last several years, various optimality and duality results have been obtained for multiobjective fractional programming problems. In Chen [1], multiobjective fractional problem and its duality theorems have been considered under higher-order \((F, \alpha, \rho, d)\)-convexity. Later on, Suneja et al. [10] discussed higher-order Mond-Weir and Schaible type nondifferentiable dual programs and their duality theorems under higher-order \((F, \rho, \sigma)\)-type \(I\)-assumptions. Recently, Ying [12]
has studied higher-order multiobjective symmetric fractional problem and formulated its Mond-Weir type dual. Further, duality results are obtained under higher-order \((F, \alpha, \rho, \delta)\)-convexity.

In this paper, we introduce a pair of nondifferentiable multiobjective Mond-Weir type higher-order symmetric fractional programming problems over cones. For a differentiable function \(h : R^n \times R^n \rightarrow R\), we introduce the definition of higher-order \((C, \alpha, \rho, \delta)\)-convexity, which extends some kinds of generalized convexity. Under the higher-order \((C, \alpha, \rho, \delta)\)-convexity assumptions, we prove the higher-order weak, strong and strict converse duality theorems.

2. PRELIMINARIES AND NOTATIONS

Let \(R^n\) be the \(n\)-dimensional Euclidean space and \(R^n_+\) be its non-negative orthant. The following conventions for vectors in \(R^n\) will be used:

\[
\begin{align*}
& x < y \quad \text{if and only if} \quad x_i < y_i, \ i = 1, 2, ..., n, \\
& x \leq y \quad \text{if and only if} \quad x_i \leq y_i, \ i = 1, 2, ..., n, \\
& x \leq y \quad \text{if and only if} \quad x_i \leq y_i, \ i = 1, 2, ..., n \text{ but } x \neq y, \\
& x \not\leq y \quad \text{is the negation of } \ x \leq y.
\end{align*}
\]

For a real-valued twice differentiable function \(h(x, y)\) defined on an open set in \(R^n \times R^n\), denote by \(\nabla_x h(\bar{x}, \bar{y})\) — the gradient vector of \(h\) with respect to \(x\) at \((\bar{x}, \bar{y})\), \(\nabla_{xy} h(\bar{x}, \bar{y})\) — the Hessian matrix with respect to \(x\) at \((\bar{x}, \bar{y})\). Similarly, \(\nabla_y h(\bar{x}, \bar{y})\), \(\nabla_{yx} h(\bar{x}, \bar{y})\) and \(\nabla_{yy} h(\bar{x}, \bar{y})\) are also defined.

**Definition 2.1** [4]. Let \(C\) be a compact convex set in \(R^n\). The support function of \(C\) is defined by

\[
s(x|C) = \max\{x^T y : y \in C\}.
\]

A support function, being convex and everywhere finite, has a subdifferential, that is, there exists a \(z \in R^n\) such that

\[
s(y|C) \geq s(x|C) + z^T(y - x), \forall x \in C.
\]

The subdifferential of \(s(x|C)\) is given by

\[
\partial s(x|C) = \{z \in C : z^T x = s(x|C)\}.
\]

For a convex set \(D \subset R^n\), the normal cone to \(D\) at a point \(x \in D\) is defined by

\[
N_D(x) = \{y \in R^n : y^T(z - x) \leq 0, \forall z \in D\}.
\]

When \(C\) is a compact convex set, \(y \in N_C(x)\) if and only if \(s(y|C) = x^T y\), or equivalently, \(x \in \partial s(y|C)\).

**Definition 2.2** [4]. The positive polar cone \(S^*\) of a cone \(S \subset R^n\) is defined by

\[
S^* = \{y \in R^n : x^T y \geq 0, \forall x \in S\}.
\]

A general multiobjective programming problem can be expressed in the following form:

\[\text{(P)} \quad \text{Minimize } f(x) = (f_1(x), f_2(x), ..., f_k(x))^T\]

subject to \(x \in X^0 = \{x \in X : g(x) \leq 0\}\),

where \(X \subset R^n\) is open, \(f : X \rightarrow R^k\), \(g : X \rightarrow R^m\), are differentiable on \(X\).
Consider the following multiobjective fractional symmetric dual programs:

\[ \text{Maximize} \sum_{i} \quad \text{subject to} \quad \]

Definition 2.3 [3]. A feasible solution \( \bar{x} \in X^0 \) is said to be a weakly efficient solution of \((P)\) if there exists no \( x \in X^0 \) such that \( f(x) < f(\bar{x}) \).

Definition 2.4 [3]. A feasible solution \( \bar{x} \in X^0 \) is said to be an efficient (or Pareto optimal) solution of \((P)\) if there exists no \( x \in X^0 \) such that \( f(x) \leq f(\bar{x}) \).

Definition 2.5 [7]. Let \( C: X \times X \times R^n \to R \ (X \subseteq R^n) \) be a function which satisfies \( C_{x,u}(0) = 0, \forall(x, u) \in X \times X \). Then, the function \( C \) is said to be convex on \( R^n \) with respect to the third argument iff for any fixed \( (x, u) \in X \times X \),

\[ C_{x,u}(\lambda x_1 + (1-\lambda)x_2) \leq \lambda C_{x,u}(x_1) + (1-\lambda)C_{x,u}(x_2), \forall \lambda \in (0, 1), \forall x_1, x_2 \in R^n. \]

Now, we introduce the definition of higher-order \((C, \alpha, \rho, d)\)-convex function:

Definition 2.6 A differentiable function \( f: X \to R \) is said to be higher order \((C, \alpha, \rho, d)\)-convex at \( u \in X \) with respect to \( h: X \times R^n \to R \) if for all \( x \in X \) and \( p \in R^n, \exists \rho \in R, \) a real valued function \( \alpha: X \times X \to R_+ \setminus \{0\} \) and \( d: X \times X \to R \) (satisfying \( d(x, z) = 0 \iff x = z \)) such that

\[ \frac{1}{\alpha(x, u)} \left[ f(x) - f(u) - h(u, p) + p^T \nabla_p h(u, p) - \rho d^2(x, u) \right] \geq C_{x,u} \left[ \nabla_x f(u) + \nabla_p h(u, p) \right]. \]

The function \( f \) is higher-order \((C, \alpha, \rho, d)\)-convex over \( X \) if, \( \forall u \in X \), it is higher \((C, \alpha, \rho, d)\)-convex.

3. Higher-order symmetric duality

Consider the following multiobjective fractional symmetric dual programs:

**MFP** Minimize \( L(x, y, p) = (L_1(x, y, p_1), L_2(x, y, p_2), ..., L_k(x, y, p_k))^T \)

subject to

\[ -\sum_{i=1}^{k} \lambda_i \left[ (\nabla_y f_i(x, y) - z_i + \nabla_p H_i(x, y, p_i)) - L_i(x, y, p_i)(\nabla_y g_i(x, y) + r_i + \nabla_p G_i(x, y, p_i)) \right] \in C_2^*, \]

\[ y^T \left[ \sum_{i=1}^{k} \lambda_i \left[ (\nabla_y f_i(x, y) - z_i + \nabla_p H_i(x, y, p_i)) - L_i(x, y, p_i)(\nabla_y g_i(x, y) + r_i + \nabla_p G_i(x, y, p_i)) \right] \right] \geq 0, \]

\[ \lambda > 0, \lambda^T e = 1, x \in C_1, z_i \in D_i, r_i \in F_i, i = 1, 2, ..., k. \]

**MFD** Maximize \( M(u, v, q) = (M_1(u, v, q_1), M_2(u, v, q_2), ..., M_k(u, v, q_k))^T \)

subject to

\[ \sum_{i=1}^{k} \lambda_i \left[ (\nabla_x f_i(u, v) + w_i + \nabla_q \Phi_i(u, v, q_i)) - M_i(u, v, q_i)(\nabla_x g_i(u, v) - t_i + \nabla_q \Psi_i(u, v, q_i)) \right] \in C_1^*, \]

\[ u^T \left[ \sum_{i=1}^{k} \lambda_i \left[ (\nabla_x f_i(u, v) + w_i + \nabla_q \Phi_i(u, v, q_i)) - M_i(u, v, q_i)(\nabla_x g_i(u, v) - t_i + \nabla_q \Psi_i(u, v, q_i)) \right] \right] \leq 0, \]

\[ \lambda > 0, \lambda^T e = 1, v \in C_2, w_i \in Q_i, t_i \in E_i, i = 1, 2, ..., k. \]
where

\[ L_i(x, y, p_i) = \frac{f_i(x, y) + s(x)q_i - y^T z_i + H_i(x, y, p_i) - p_i^T \nabla_p H_i(x, y, p_i)}{g_i(x, y) - s(x)E_i + y^T r_i + G_i(x, y, p_i) - p_i^T \nabla_p G_i(x, y, p_i)}, \]

\[ M_i(u, v, q_i) = \frac{f_i(u, v) - s(v)d_i + u^T w_i + \Phi_i(u, v, q_i) - q_i^T \nabla_q \Phi_i(u, v, q_i)}{g_i(u, v) + s(v)F_i - u^T t_i + \Psi_i(u, v, q_i) - q_i^T \nabla_q \Psi_i(u, v, q_i)}, \]

where \( f_i : S_1 \times S_2 \to R \); \( g_i : S_1 \times S_2 \to R \); \( H_i, G_i : S_1 \times S_2 \times R^m \to R \) and \( \Phi_i, \Psi_i : S_1 \times S_2 \times R^m \to R \) are differentiable functions for all \( i = 1, 2, ..., k \). \( S_1 \subseteq R^m \) and \( S_2 \subseteq R^m \) are such that \( C_1 \times C_2 \subset S_1 \times S_2 \), where \( C_1 \) and \( C_2 \) are the closed convex cones in \( R^m \) and \( R^m \), respectively. \( Q_i, E_i \) are compact convex sets in \( R^m \) and \( D_i, F_i \) are compact convex sets in \( R^m \), \( e = (1, 1, ..., 1)^T \in R^k \), \( p_i \in R^m \), \( q_i \in R^m \), \( i = 1, 2, ..., k \), \( p = (p_1, p_2, ..., p_k) \), \( q = (q_1, q_2, ..., q_k) \). \( C_1^* \) and \( C_2^* \) are positive polar cones of \( C_1 \) and \( C_2 \), respectively. It is assumed that in the feasible regions, the numerators are nonnegative and denominators are positive.

Let \( U = (U_1, U_2, ..., U_k)^T \) and \( V = (V_1, V_2, ..., V_k)^T \). Then, we can express the programs (MFP) and (MFD) equivalently as:

\[ \textbf{(MFP)}_U \text{ Minimize } U \text{ subject to } \]

\[ (f_i(x, y) + s(x)q_i - y^T z_i + H_i(x, y, p_i) - p_i^T \nabla_p H_i(x, y, p_i)) - U_i(g_i(x, y) - s(x)E_i + y^T r_i + G_i(x, y, p_i) - p_i^T \nabla_p G_i(x, y, p_i)) = 0, \quad i = 1, 2, ..., k. \]  

\[ \sum_{i=1}^{k} \lambda_i \left[ \nabla_y f_i(x, y) - z_i + \nabla_p H_i(x, y, p_i) \right] - U_i(\nabla_y g_i(x, y) + r_i + \nabla_p G_i(x, y, p_i)) \in C_2^*, \]

\[ y^T \left[ \sum_{i=1}^{k} \lambda_i \left[ \nabla_y f_i(x, y) - z_i + \nabla_p H_i(x, y, p_i) \right] - U_i(\nabla_y g_i(x, y) + r_i + \nabla_p G_i(x, y, p_i)) \right] \geq 0, \]

\[ \lambda > 0, \quad \lambda^T e = 1, \quad x \in C_1, \quad z_i \in D_i, \quad r_i \in F_i, \quad i = 1, 2, ..., k. \]

\[ \textbf{(MFD)}_V \text{ Maximize } V \text{ subject to } \]

\[ (f_i(u, v) - s(v)d_i + u^T w_i + \Phi_i(u, v, q_i) - q_i^T \nabla_q \Phi_i(u, v, q_i)) - V_i(g_i(u, v) + s(v)F_i - u^T t_i + \Psi_i(u, v, q_i) - q_i^T \nabla_q \Psi_i(u, v, q_i)) = 0, \quad i = 1, 2, ..., k. \]  

\[ \sum_{i=1}^{k} \lambda_i \left[ \nabla_x f_i(u, v) + w_i + \nabla_q \Phi_i(u, v, q_i) \right] - V_i(\nabla_x g_i(u, v) - t_i + \nabla_q \Psi_i(u, v, q_i)) \in C_1^*, \]
Next, we prove weak, strong and converse duality theorems for (MFP) and (MFD), which therefore are equally applicable for (MFP) and (MFD). Let \( z = (z_1, z_2, ..., z_k) \), \( r = (r_1, r_2, ..., r_k) \), \( w = (w_1, w_2, ..., w_k) \) and \( t = (t_1, t_2, ..., t_k) \).

**Theorem 3.1.** (Weak duality). Let \((x, y, U, z, r, \lambda, p)\) be feasible for (MFP)\(_U\) and let \((u, v, V, w, t, \lambda, q)\) be feasible for (MFD)\(_V\). Let \( \forall i \in \{1, 2, ..., k\}, f_i(., v) + (.)^T w_i \) be higher order \((C, \alpha, \rho_i, d_i)\) convex at \( u \) with respect to \( \Phi_i(u, v, q_i) \), \(- (g_i(., v) - (.)^T t_i) \) be higher order \((C, \alpha, \rho_i, d_i)\) convex at \( u \) with respect to \(- \Psi_i(u, v, q_i) \), and \((g_i(., v) + (.)^T t_i) \) be higher order \((C, \alpha, \rho_i, d_i)\) convex at \( y \) with respect to \( H_i(x, y, p_i) \) and \((g_i(., v) + (.)^T t_i) \) be higher order \((C, \alpha, \rho_i, d_i)\) convex at \( y \) with respect to \( G_i(x, y, p) \) where \( C : R^n \times R^n \times R^m \rightarrow R \) and \( \tilde{C} : R^n \times R^n \times R^m \rightarrow R \). If the following conditions hold:

\[
g_i(x, v) + v^T r_i - x^T t_i > 0, \quad i = 1, 2, ..., k, \tag{3.7}
\]

or

\[
either \sum_{i=1}^{k} \lambda_i [(1 + V_i)\rho_i d_i^2(x, u) + (1 + U_i)\bar{\rho}_i \tilde{d}_i^2(v, y)] \geq 0 \quad \text{or} \quad \rho_i \geq 0 \quad i = 1, 2, ..., k. \tag{3.8}
\]

\[
C_{x,u}(a) + a^T u \geq 0, \quad \forall a \in C_1^*, \quad \tilde{C}_{v,y}(b) + b^T y \geq 0, \quad \forall b \in C_2^*. \tag{3.9}
\]

Then, \( U \not\in V \).

**Proof.** Since \( \forall i \in \{1, 2, ..., k\}, f_i(., v) + (.)^T w_i \) and \(- (g_i(., v) - (.)^T t_i) \) is higher-order \((C, \alpha, \rho_i, d_i)\) convex in the first variable at \( u \) for fixed \( v \), we have

\[
\frac{1}{\alpha(x, u)} \left[ f_i(x, v) + x^T w_i - f_i(u, v) - u^T w_i - \Phi_i(u, v, q_i) + q_i^T \nabla_q \Phi_i(u, v, q_i) \right]
\]

\[
- \rho_i d_i^2(x, u) \geq C_{x,u}(\nabla_x f_i(u, v) + w_i + \nabla_q \Phi_i(u, v, q_i)) \tag{3.10}
\]

and

\[
\frac{1}{\alpha(x, u)} \left[ -(g_i(x, v) + x^T t_i + g_i(u, v) - u^T t_i) + (\Psi_i(u, v, q_i) - q_i^T \nabla_q \Psi_i(u, v, q_i)) \right]
\]

\[
- \rho_i d_i^2(x, u) \geq C_{x,u}(\nabla_x g_i(u, v) + t_i - \nabla_q \Psi_i(u, v, q_i)). \tag{3.11}
\]

Multiplying \( \frac{\lambda_i}{\tau} > 0 \) and \( \frac{\lambda_i V_i}{\tau} \geq 0, \quad i = 1, 2, ..., k \), where \( \tau = 1 + \sum_{i=1}^{k} \lambda_i V_i \) together with (3.10) and (3.11), respectively, we obtain

\[
\frac{\lambda_i}{\alpha(x, u) \tau} \left( f_i(x, v) + x^T w_i - f_i(u, v) - u^T w_i - \Phi_i(u, v, q_i) + q_i^T \nabla_q \Phi_i(u, v, q_i) \right)
\]
\[-\frac{\lambda_i}{\alpha(x, u) \tau} \rho_i d_i^2(x, u) \geq \frac{\lambda_i}{\tau} C_{x, u} \left( \nabla_x f_i(u, v) + w_i + \nabla_q \Phi_i(u, v, q_i) \right) \]

and
\[
\frac{\lambda_i V_i}{\alpha(x, u) \tau} \left[ -g_i(x, v) + x^T t_i + g_i(u, v) - u^T t_i + \Psi_i(u, v, q_i) - q_i^T \nabla_q \Psi_i(u, v, q_i) \right]
- \frac{\lambda_i V_i}{\alpha(x, u) \tau} \rho_i d_i^2(x, u) \geq \frac{\lambda_i V_i}{\tau} C_{x, u} \left( -\nabla_x g_i(u, v) + t_i - \nabla_q \Psi_i(u, v, q_i) \right).
\]

Now, summing over \( i \) and adding the above two inequalities, using convexity of \( C \), we have

\[
\sum_{i=1}^{k} \frac{\lambda_i}{\alpha(x, u) \tau} \left[ f_i(x, v) + x^T w_i - f_i(u, v) - u^T w_i - \Phi_i(u, v, q_i) + q_i^T \nabla_q \Phi_i(u, v, q_i) \right]
+ \sum_{i=1}^{k} \frac{\lambda_i V_i}{\alpha(x, u) \tau} \left[ -g_i(x, v) + x^T t_i + g_i(u, v) - u^T t_i + \Psi_i(u, v, q_i) - q_i^T \nabla_q \Psi_i(u, v, q_i) \right]
- \sum_{i=1}^{k} \frac{\lambda_i}{\alpha(x, u) \tau} \left( 1 + V_i \right) \rho_i d_i^2(x, u) \geq C_{x, u} \left[ \sum_{i=1}^{k} \frac{\lambda_i}{\tau} \left( \nabla_x f_i(u, v) + w_i + \nabla_q \Phi_i(u, v, q_i) \right) - V_i \left( \nabla_x g_i(u, v) - t_i + \nabla_q \Psi_i(u, v, q_i) \right) \right].
\] (3.12)

Now, from (3.5), as \( \tau > 0 \), we have

\[
a = \sum_{i=1}^{k} \frac{\lambda_i}{\tau} \left[ (\nabla_x f_i(u, v) + w_i + \nabla_q \Phi_i(u, v, q_i) - V_i (\nabla_x g_i(u, v) - t_i + \nabla_q \Psi_i(u, v, q_i)) \right] \in C_1^*.
\]

Hence, for this \( a \), \( C_{x, u}(a) \geq -u^T a \geq 0 \) (from (3.9)). Using this, in (3.12), we obtain

\[
\sum_{i=1}^{k} \frac{\lambda_i}{\alpha(x, u) \tau} \left[ f_i(x, v) + x^T w_i - f_i(u, v) - u^T w_i - \Phi_i(u, v, q_i) + q_i^T \nabla_q \Phi_i(u, v, q_i) \right]
+ \sum_{i=1}^{k} \frac{\lambda_i V_i}{\alpha(x, u) \tau} \left[ -g_i(x, v) + x^T t_i + g_i(u, v) - u^T t_i + \Psi_i(u, v, q_i) - q_i^T \nabla_q \Psi_i(u, v, q_i) \right]
\geq \sum_{i=1}^{k} \frac{\lambda_i}{\alpha(x, u) \tau} \left( 1 + V_i \right) \rho_i d_i^2(x, u).
\]

Since \( v^T r_i \leq s(v | F_i) \) and using (3.4) in above inequality, we get

\[
\sum_{i=1}^{k} \lambda_i [f_i(x, v) + x^T w_i - s(v | D_i) + V_i (x^T t_i - v^T r_i - g_i(x, v))] \geq \sum_{i=1}^{k} \lambda_i (1 + V_i) \rho_i d_i^2(x, u).
\] (3.13)
Similarly, by the higher-order \((\tilde{C}, \tilde{\alpha}, \tilde{\rho}, \tilde{d})\) - convexity of \(-f_i(x,.) + (.)^T z_i\) and \((g_i(x,.) + (.)^T r_i), \forall i \in \{1, 2, ..., k\}\), in the second variable at \(y\), for fixed \(x\) and from the condition (3.9), for

\[
b = -\sum_{i=1}^{k} \frac{\lambda_i}{\tau} ((\nabla_y f_i(x, y) - z_i + \nabla_p H_i(x, y, p_i) - U_i(\nabla_y g_i(x, y) + r_i + \nabla p_i G_i(x, y, p_i))) \in C^*_2,
\]

we get

\[
\sum_{i=1}^{k} \lambda_i [-f_i(x, v) + v^T z_i - s(x|Q_i)] + U_i(v^T r_i - x^T t_i + g_i(x, v)] \geq 0. \tag{3.14}
\]

Adding the inequalities (3.13), (3.14) and applying (3.8), we get

\[
\sum_{i=1}^{k} \lambda_i (v^T z_i - s(v|D_i) + x^T w_i - s(x|Q_i))
\]

\[
+ \sum_{i=1}^{k} \lambda_i(U_i - V_i)(g_i(x, v) + v^T r_i - x^T t_i) \geq 0. \tag{3.15}
\]

Since \(\lambda > 0\), \(v^T z_i \leq s(v|D_i)\) and \(x^T w_i \leq s(x|Q_i)\), the above inequality gives

\[
\sum_{i=1}^{k} \lambda_i(U_i - V_i)(g_i(x, v) + v^T r_i - x^T t_i) \geq 0. \tag{3.16}
\]

From the fact that \(\lambda > 0\) and using (3.7), it follows that \(U \nless V\). This completes the proof. \(\square\)

**Theorem 3.2.** (Weak duality). Let \((x, y, U, z, r, \lambda, p)\) and \((u, v, W, t, \lambda, q)\) be feasible solutions of (MFP)$_U$ and (MFD)$_V$, respectively. Suppose that

(i) \((-f_i(x,.)) + (-)^T w_i - V_i(g_i(x,.)) + (-)^T r_i) is higher-order \((C, \alpha, \rho, d)\) - convex at \(u\) with respect to \((\Phi_i(u, v, q_i) - V_i(\Psi_i(u, v, q_i))\),

(ii) \((-f_i(x,.)) + (-)^T z_i) + U_i(g_i(x,.)) + (-)^T r_i) is higher-order \((C, \alpha, \rho, d)\) - convex at \(y\) with respect to \(-H_i(x, y, p) + U_i G_i(x, y, p)\),

(iii) either \(\sum_{i=1}^{k} \lambda_i [p_i d_i^2(x, u) + \tilde{\rho}_i d_i^2(v, y)] \geq 0\) or \(\rho_i \geq 0\) and \(\tilde{\rho}_i \geq 0, i = 1, 2, \ldots, k\),

(iv) \(C_{x,u}(a) + a^T u \geq 0, \forall a \in C^*_1, C_{u,v}(b) + b^T y \geq 0, \forall b \in C^*_2\),

Then, \(U \nless V\).

**Proof.** By hypothesis (i), we have

\[
\frac{1}{\alpha(x, u)} \left[ f_i(x, u) + x^T w_i - V_i(g_i(x, v) - x^T t_i) - (f_i(u, v) + u^T w_i) - V_i(g_i(u, v) - u^T t_i) - (\Phi_i(u, v, q_i) - V_i(\Psi_i(u, v, q_i)) + q_i^T (\nabla_q \Phi_i(u, v, q_i) - V_i \nabla_q \Psi_i(u, v, q_i)) - \rho_i d_i^2(x, u) \right]
\]

\[
\geq C_{x,u} [\nabla_x f_i(u, v) + w_i - V_i(\nabla_x g_i(u, v) - t_i) + (\nabla_q \Phi_i(u, v, q_i) - V_i \nabla_q \Psi_i(u, v, q_i))].
\]
Since $\lambda > 0$, we obtain
\[
\sum_{i=1}^{k} \frac{\lambda_i}{\alpha(x,u)} [f_i(x,v) + x^T w_i - f_i(u,v) - u^T w_i - \Phi_i(u,v,q_i) + q_i^T \nabla_q \Phi_i(u,v,q_i)] + V_i(-g_i(x,v) + x^T t_i + g_i(u,v) - u^T t_i + \Psi_i(u,v,q_i) - q_i^T \nabla_q \Psi_i(u,v,q_i)) - \rho_i d_i^2(x,u)
\]
\[
\geq \sum_{i=1}^{k} \lambda_i C_{x,u} \left[ (\nabla_x f_i(u,v) + w_i - V_i(\nabla_x g_i(u,v) - t_i) + (\nabla_q \Phi_i(u,v,q_i) - V_i(\nabla_q \Psi_i(u,v,q_i))] \right].
\]

Using convexity of $C$, we have
\[
\frac{1}{\alpha(x,u)} \left[ \sum_{i=1}^{k} \lambda_i \left( f_i(x,v) + x^T w_i - f_i(u,v) - u^T w_i - \Phi_i(u,v,q_i) + q_i^T \nabla_q \Phi_i(u,v,q_i) \right) 
\right.
\]
\[
+ \sum_{i=1}^{k} \lambda_i V_i(-g_i(x,v) + x^T t_i - u^T r_i) + \sum_{i=1}^{k} \lambda_i V_i(g_i(u,v) + v^T r_i - u^T t_i + \Psi_i(u,v,q_i) - q_i^T \nabla_q \Psi_i(u,v,q_i)) - \frac{1}{\alpha(x,u)} \sum_{i=1}^{k} \lambda_i \rho_i d_i^2(x,u)
\]
\[
\geq C_{x,u} \left[ \sum_{i=1}^{k} \lambda_i \left( (\nabla_x f_i(u,v) + w_i + \nabla_q \Phi_i(u,v,q_i) - V_i(\nabla_x g_i(u,v) - t_i + \nabla_q \Psi_i(u,v,q_i)) \right) ].
\]

From (3.5),
\[
a = \sum_{i=1}^{k} \lambda_i [\nabla_x f_i(u,v) + w_i + \nabla_q \Phi_i(u,v,q_i) - V_i(\nabla_x g_i(u,v) - t_i + \nabla_q \Psi_i(u,v,q_i))] \in C^*_1
\]

and using hypothesis $(iv)$, we get
\[
\frac{1}{\alpha(x,u)} \left[ \sum_{i=1}^{k} \lambda_i \left( f_i(x,v) + x^T w_i - f_i(u,v) - u^T w_i - \Phi_i(u,v,q_i) + q_i^T \nabla_q \Phi_i(u,v,q_i) \right) 
\right.
\]
\[
+ \sum_{i=1}^{k} \lambda_i V_i(-g_i(x,v) + x^T t_i - u^T r_i) + \sum_{i=1}^{k} \lambda_i V_i(g_i(u,v) - u^T t_i + \Psi_i(u,v,q_i) - q_i^T \nabla_q \Psi_i(u,v,q_i)) \geq \sum_{i=1}^{k} \lambda_i \frac{\rho_i d_i^2(x,u).}{\alpha(x,u)}.
\]

Since $v^T r_i \leq s(v|F_i)$, using (3.4) in (3.17), we get
\[
\sum_{i=1}^{k} \lambda_i \left( f_i(x,v) + x^T w_i - s(v|D_i) + V_i(x^T t_i - v^T r_i - g_i(x,v)) \right)
\]
\[
\geq \sum_{i=1}^{k} \lambda_i \rho_i d_i^2(x,u). (3.18)
\]

Similarly, by the higher-order $(\bar{C}, \bar{\alpha}, \bar{\rho}, \bar{d})$-convexity of $-f_i(x,) + (.)^T z_i + U_i(g_i(x,)) + (.)^T r_i$ in the second variable at $y$, for fixed $x$, we get

\[
\sum_{i=1}^{k} \lambda_i \left( f_i(x,v) + x^T w_i - s(v|D_i) + V_i(x^T t_i - v^T r_i - g_i(x,v)) \right)
\]
\[
\geq \sum_{i=1}^{k} \lambda_i \rho_i d_i^2(x,u). (3.18)
\]
\[
\sum_{i=1}^{k} \lambda_i [-f_i(x, v) + v^T z_i - s(x|Q_i) + U_i(v^T r_i - x^T t_i + g_i(x, v))]
\geq \sum_{i=1}^{k} \lambda_i \bar{\rho}_i d_i^2(v, y).
\]  
(3.19)

Using \( \lambda > 0 \), \( v^T z_i \leq s(v|D_i) \) and \( x^T w_i \leq s(x|Q_i) \), it follows from (3.18) and (3.19) that

\[
\sum_{i=1}^{k} \lambda_i (U_i - V_i)(g_i(x, v) + v^T r_i - x^T t_i) \geq 0.
\]  
(3.20)

Since \( \lambda > 0 \) and using hypothesis \( (v) \), it follows that \( U \not\in V \). Hence, the result. \( \square \)

**Theorem 3.3. (Strong duality).** Let \((\bar{x}, \bar{y}, \bar{U}, \bar{z}, \bar{r}, \bar{\lambda}, \bar{p})\) be an efficient solution of \((MFP)_U\) and fix \( \lambda = \bar{\lambda} \) in \((MFD)_V\). If the following conditions hold

(i) \( \nabla_x H_i(\bar{x}, \bar{y}, 0) = \nabla_x G_i(\bar{x}, \bar{y}, 0) = 0, \nabla_q P_i(\bar{x}, \bar{y}, 0) = \nabla_q P_i(\bar{x}, \bar{y}, 0) = 0, H_i(\bar{x}, \bar{y}, 0) = G_i(\bar{x}, \bar{y}, 0) = 0, P_i(\bar{x}, \bar{y}, 0) = P_i(\bar{x}, \bar{y}, 0) = 0, \nabla_p H_i(\bar{x}, \bar{y}, 0) = \nabla_p G_i(\bar{x}, \bar{y}, 0) = 0, i = 1, 2, ..., k, \)

(ii) for all \( i \in \{1, 2, ..., k\} \), the Hessian matrix \( \nabla_{v,p} H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{U}_i \nabla_{v,p} G_i(\bar{x}, \bar{y}, \bar{p}_i) \) is positive or negative definite,

(iii) the set of vectors \( \{\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_y H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{U}_i(\nabla_y g_i(\bar{x}, \bar{y}) + \bar{r}_i + \nabla_y G_i(\bar{x}, \bar{y}, \bar{p}_i))\}\)\( \subseteq \mathbb{R}^n \) is linearly independent.

(iv) for \( \bar{p}_i \in \mathbb{R}^n, \bar{p}_i \neq 0 (i = 1, 2, ..., k) \) implies that

\[ \sum_{i=1}^{k} \bar{p}_i^T [\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_y H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{U}_i(\nabla_y g_i(\bar{x}, \bar{y}) + \bar{r}_i + \nabla_y G_i(\bar{x}, \bar{y}, \bar{p}_i))] \neq 0. \]

(v) \( \bar{U}_i > 0, \forall i \in \{1, 2, ..., k\} \).

Then

(a) \( \bar{p}_i = 0, i = 1, 2, ..., k, \)

(b) there exists \( \bar{w}_i \in Q_i \) and \( \bar{t}_i \in E_i, i = 1, 2, ..., k \) such that \( (\bar{x}, \bar{y}, \bar{U}, \bar{w}, \bar{t}, \bar{\lambda}, \bar{q} = 0) \) is a feasible solution of \((MFD)_V\).

Furthermore, if the hypotheses in Theorem 3.1 or 3.2 are satisfied, then \((\bar{x}, \bar{y}, \bar{U}, \bar{z}, \bar{r}, \bar{\lambda}, \bar{p})\) is an efficient solution of \((MFD)_V\), and the two objective values are equal.

**Proof.** Since \((\bar{x}, \bar{y}, \bar{U}, \bar{z}, \bar{r}, \bar{\lambda}, \bar{p})\) is an efficient solution of \((MFP)_U\), therefore, by the Fritz John necessary optimality conditions [2], there exists \( \alpha \in \mathbb{R}^k, \beta \in \mathbb{R}^k, \gamma \in C_2, \delta \in R_+, \xi \in \mathbb{R}^k, \eta \in R, \bar{w}_i \in \mathbb{R}^n \) and \( \bar{t}_i \in \mathbb{R}^n, i = 1, 2, ..., k \) such that

\[
(x - \bar{x})^T \sum_{i=1}^{k} \beta_i (\nabla_x f_i(\bar{x}, \bar{y}) + \bar{w}_i + \nabla_x H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{U}_i(\nabla_y g_i(\bar{x}, \bar{y}) + \bar{r}_i + \nabla_y G_i(\bar{x}, \bar{y}, \bar{p}_i))) +
\]

\[
(\gamma - \delta \bar{y})^T \sum_{i=1}^{k} \lambda_i (\nabla_y f_i(\bar{x}, \bar{y}) - \bar{U}_i(\nabla_y g_i(\bar{x}, \bar{y}) + \bar{r}_i + \nabla_y G_i(\bar{x}, \bar{y}, \bar{p}_i)) - \bar{U}_i(\nabla_{p,x} G_i(\bar{x}, \bar{y}, \bar{p}_i))^T (\gamma - \delta \bar{y})_i) \geq 0, \forall x \in C_1,
\]  
(3.21)

\[
\sum_{i=1}^{k} \beta_i (\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_y H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{U}_i(\nabla_y g_i(\bar{x}, \bar{y}) + \bar{r}_i + \nabla_y G_i(\bar{x}, \bar{y}, \bar{p}_i)))
\]
\[ \begin{align*}
&+ \sum_{i=1}^{k} \bar{\lambda}_i (\nabla_{yy} f_i(x, \bar{y}) - \bar{U}_i \nabla_{yy} g_i(x, \bar{y}))^T (\gamma - \delta \bar{y}) + \sum_{i=1}^{k} (\nabla_{p_i} H_i(x, \bar{y}, \bar{p}_i)) \\
&- \bar{U}_i \nabla_{p_i} G_i(x, \bar{y}, \bar{p}_i))^T (-\beta_i \bar{p}_i + (\gamma - \delta \bar{y}) \bar{\lambda}_i) - \delta \sum_{i=1}^{k} \bar{\lambda}_i [\nabla_{y} f_i(x, \bar{y}) - \bar{z}_i \\
&+ \nabla_{p_i} H_i(x, \bar{y}, \bar{p}_i) - \bar{U}_i (\nabla_{yy} g_i(x, \bar{y}) + \bar{r}_i + \nabla_{p_i} G_i(x, \bar{y}, \bar{p}_i)) = 0, \quad \text{(3.22)}
\end{align*} \]

\[ \begin{align*}
\alpha_i - \beta_i (g_i(x, \bar{y}) - s(x|E_i) + \bar{y}^T \bar{r}_i + G_i(x, \bar{y}, \bar{p}_i) - \bar{p}_i^T \nabla_{p_i} G_i(x, \bar{y}, \bar{p}_i)) \\
- (\gamma - \delta \bar{y})^T \nabla_{y} (\bar{\lambda}_i (g_i(x, \bar{y}) + \bar{r}_i + \nabla_{p_i} G_i(x, \bar{y}, \bar{p}_i))) = 0, \quad i = 1, 2, \ldots, k, \quad \text{(3.23)}
\end{align*} \]

\[ \begin{align*}
(\gamma - \delta \bar{y})^T (\nabla_{y} f_i(x, \bar{y}) - \bar{z}_i + \nabla_{p_i} H_i(x, \bar{y}, \bar{p}_i)) \\
- \bar{U}_i (\nabla_{yy} g_i(x, \bar{y}) + \bar{r}_i + \nabla_{p_i} G_i(x, \bar{y}, \bar{p}_i)) - \xi_i + \eta = 0, \quad i = 1, 2, \ldots, k, \quad \text{(3.24)}
\end{align*} \]

\[ \begin{align*}
\bar{\lambda}_i (\gamma - \delta \bar{y}) - \beta_i \bar{p}_i)^T (\nabla_{p_i} H_i(x, \bar{y}, \bar{p}_i) \\
- \bar{U}_i (\nabla_{p_i} G_i(x, \bar{y}, \bar{p}_i)) = 0, \quad i = 1, 2, \ldots, k, \quad \text{(3.25)}
\end{align*} \]

\[ \begin{align*}
\beta_i \bar{y} + (\gamma - \delta \bar{y}) \bar{\lambda}_i \in N_{D_i}(\bar{z}_i), \quad i = 1, 2, \ldots, k, \quad \text{(3.26)}
\end{align*} \]

\[ \begin{align*}
\beta_i \bar{U}_i \bar{y} + \bar{\lambda}_i \bar{U}_i (\gamma - \delta \bar{y}) \in N_{F_i}(\bar{r}_i), \quad i = 1, 2, \ldots, k, \quad \text{(3.27)}
\end{align*} \]

\[ \begin{align*}
\gamma^T \sum_{i=1}^{k} \bar{\lambda}_i ((\nabla_{y} f_i(x, \bar{y}) - \bar{z}_i + \nabla_{p_i} H_i(x, \bar{y}, \bar{p}_i)) \\
- \bar{U}_i (\nabla_{yy} g_i(x, \bar{y}) + \bar{r}_i + \nabla_{p_i} G_i(x, \bar{y}, \bar{p}_i)) = 0, \quad \text{(3.28)}
\end{align*} \]

\[ \begin{align*}
\delta \bar{y}^T \sum_{i=1}^{k} \bar{\lambda}_i ((\nabla_{y} f_i(x, \bar{y}) - \bar{z}_i + \nabla_{p_i} H_i(x, \bar{y}, \bar{p}_i)) \\
- \bar{U}_i (\nabla_{yy} g_i(x, \bar{y}) + \bar{r}_i + \nabla_{p_i} G_i(x, \bar{y}, \bar{p}_i)) = 0, \quad \text{(3.29)}
\end{align*} \]

\[ \begin{align*}
\bar{\lambda}_i^T \xi = 0, \quad \text{(3.30)}
\end{align*} \]

\[ \begin{align*}
\eta (\bar{\lambda}_i^T e - 1) = 0, \quad \text{(3.31)}
\end{align*} \]

\[ \begin{align*}
\bar{w}_i \in Q_i, \bar{t}_i \in E_i, \bar{x}^T t_i = s(x|E_i), \bar{x}^T \bar{w}_i = s(x|Q_i), \quad i = 1, 2, \ldots, k, \quad \text{(3.32)}
\end{align*} \]

\[ \begin{align*}
(\alpha, \delta, \xi) \geq 0, \quad (\alpha, \beta, \gamma, \delta, \xi, \eta) \neq 0. \quad \text{(3.33)}
\end{align*} \]

Since \( \bar{\lambda} > 0 \) and \( \xi \geq 0 \), (30) implies that \( \xi = 0 \).

Equation (3.22) can be re-written as

\[ \begin{align*}
\sum_{i=1}^{k} (\beta_i - \delta \bar{\lambda}_i)((\nabla_{y} f_i(x, \bar{y}) - \bar{z}_i + \nabla_{p_i} H_i(x, \bar{y}, \bar{p}_i)) - \bar{U}_i (\nabla_{yy} g_i(x, \bar{y}) + \bar{r}_i + \nabla_{p_i} G_i(x, \bar{y}, \bar{p}_i)) \\
+ \sum_{i=1}^{k} \beta_i ((\nabla_{y} H_i(x, \bar{y}, \bar{p}_i) - \bar{U}_i \nabla_{yy} G_i(x, \bar{y}, \bar{p}_i)) - (\nabla_{p_i} H_i(x, \bar{y}, \bar{p}_i) - \bar{U}_i \nabla_{p_i} G_i(x, \bar{y}, \bar{p}_i)) \\
+ \sum_{i=1}^{k} \bar{\lambda}_i ((\nabla_{yy} f_i(x, \bar{y}) - \bar{U}_i \nabla_{yy} g_i(x, \bar{y}))^T (\gamma - \delta \bar{y}) \\
+ \sum_{i=1}^{k} (\nabla_{p_i} H_i(x, \bar{y}, \bar{p}_i) - \bar{U}_i \nabla_{p_i} G_i(x, \bar{y}, \bar{p}_i))^T (-\beta_i \bar{p}_i + (\gamma - \delta \bar{y}) \bar{\lambda}_i) = 0. \quad \text{(3.34)}
\end{align*} \]
By hypothesis \((ii)\) and (3.25), we have
\[ \lambda_i(\gamma - \delta \bar{y}) = \beta_i \bar{p}_i, \quad i = 1, 2, ..., k. \] (3.35)

Now, we claim that \(\beta_i \neq 0, \quad \forall i\). If possible, let \(\beta_{t_0} = 0\) for some \(t_0, 1 \leq t_0 \leq k\), then from \(\lambda_{t_0} > 0\) and equation (3.35), we have
\[ \gamma = \delta \bar{y}. \] (3.36)

Using (3.35) and (3.36), we obtain \(\beta_i \bar{p}_i = 0, \quad i = 1, 2, ..., k\). Hence, by hypothesis \((i)\), we get
\[
\sum_{i=1}^{k} \beta_i ((\nabla_y H_i(\bar{x}, \bar{y}, \bar{p}_i)) - \bar{U}_i \nabla_y G_i(\bar{x}, \bar{y}, \bar{p}_i)) = 0. \] (3.37)

Using (3.35)-(3.37) in (3.34), we obtain
\[
\sum_{i=1}^{k} (\beta_i - \delta \lambda_i)((\nabla_y h_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_p H_i(\bar{x}, \bar{y}, \bar{p}_i)) - \bar{U}_i(\nabla_y g_i(\bar{x}, \bar{y}) + \bar{r}_i + \nabla_p G_i(\bar{x}, \bar{y}, \bar{p}_i))) = 0, \] (3.38)

which by hypothesis \((iii)\), follows that
\[ \beta_i - \delta \lambda_i = 0, \quad i = 1, 2, ..., k. \] (3.39)

Now, for \(i = t_0\), we have \(\delta \lambda_{t_0} = 0\). This implies \(\delta = 0\) as \(\lambda > 0\). Hence, from (3.39), \(\beta_i = 0, \quad \forall i\). Thus, from relation (3.23) and (3.36), we get \(\alpha_i = 0, \quad i = 1, 2, ..., k\).

Also, from relations (3.24) and (3.36), we get \(\eta = 0\) and \(\gamma = 0\), respectively, which contradicts the fact that \((\alpha, \beta, \gamma, \delta, \xi, \eta) \neq 0\). Hence \(\beta_i \neq 0, \quad i = 1, 2, ..., k\).

Now, equation (3.24) reduces to
\[
(\gamma - \delta \bar{y})^T (\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_p H_i(\bar{x}, \bar{y}, \bar{p}_i)) - \bar{U}_i(\nabla_y g_i(\bar{x}, \bar{y}) + \bar{r}_i + \nabla_p G_i(\bar{x}, \bar{y}, \bar{p}_i)) + \eta = 0, \quad i = 1, 2, ..., k. \] (3.40)

Multiplying by \(\lambda_i\) and summing over \(i\), we get
\[
(\gamma - \delta \bar{y})^T \sum_{i=1}^{k} \lambda_i ((\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_p H_i(\bar{x}, \bar{y}, \bar{p}_i)) - \bar{U}_i(\nabla_y g_i(\bar{x}, \bar{y}) + \bar{r}_i + \nabla_p G_i(\bar{x}, \bar{y}, \bar{p}_i)) + \eta \lambda^T e_k = 0. \] (3.41)

Subtracting (3.28) and (3.29), we get
\[
(\gamma - \delta \bar{y})^T \sum_{i=1}^{k} \lambda_i ((\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_p H_i(\bar{x}, \bar{y}, \bar{p}_i)) - \bar{U}_i(\nabla_y g_i(\bar{x}, \bar{y}) + \bar{r}_i + \nabla_p G_i(\bar{x}, \bar{y}, \bar{p}_i)) = 0. \] (3.42)

Using (3.42) in (3.41), we get, \(\eta = 0\).

Now, equation (3.40), yield
\[
(\gamma - \delta \bar{y})^T (\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_p H_i(\bar{x}, \bar{y}, \bar{p}_i)) - \bar{U}_i(\nabla_y g_i(\bar{x}, \bar{y}) + \bar{r}_i + \nabla_p G_i(\bar{x}, \bar{y}, \bar{p}_i)) = 0, \quad i = 1, 2, ..., k. \] (3.43)
Since $\bar{\lambda} > 0$, using (3.35) in (3.43), we get
\[
\beta_i p_i^T [\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_p H_i(\bar{x}, \bar{y}, \bar{p}_i)] - \bar{U}_i(\nabla_y g_i(\bar{x}, \bar{y}) + \bar{r}_i + \nabla_p G_i(\bar{x}, \bar{y}, \bar{p}_i))] = 0, \quad i = 1, 2, ..., k, \tag{3.44}
\]
Since $\beta_i \neq 0$, $i = 1, 2, ..., k$, we obtain
\[
\bar{p}_i^T [\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_p H_i(\bar{x}, \bar{y}, \bar{p}_i)] - \bar{U}_i(\nabla_y g_i(\bar{x}, \bar{y}) + \bar{r}_i + \nabla_p G_i(\bar{x}, \bar{y}, \bar{p}_i))] = 0, \quad i = 1, 2, ..., k, \tag{3.45}
\]
or
\[
\sum_{i=1}^k \bar{p}_i^T [\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_p H_i(\bar{x}, \bar{y}, \bar{p}_i)] - \bar{U}_i(\nabla_y g_i(\bar{x}, \bar{y}) + \bar{r}_i + \nabla_p G_i(\bar{x}, \bar{y}, \bar{p}_i))] = 0. \tag{3.46}
\]
By the hypothesis (iv), we have $\bar{p}_i = 0$, $i = 1, 2, ..., k$. Further using, hypothesis (i), (3.35) (3.36) in (3.21) and (3.34), respectively, we get
\[
(x - \bar{x})^T \left[ \sum_{i=1}^k \beta_i (\nabla_y f_i(\bar{x}, \bar{y}) + \bar{w}_i - \bar{U}_i(\nabla_x g_i(\bar{x}, \bar{y}) - \bar{t}_i)) \right] \geq 0, \quad \forall x \in C_1. \tag{3.47}
\]
\[
\sum_{i=1}^k (\beta_i - \delta \bar{\lambda}_i) [\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i - \nabla_p G_i(\bar{x}, \bar{y}, \bar{p}_i)] = 0. \tag{3.48}
\]
Using hypothesis (iii) in (3.48), we have
\[
\beta_i = \delta \bar{\lambda}_i, \quad i = 1, 2, ..., k. \tag{3.49}
\]
Since $\beta_i \neq 0$, $\bar{\lambda}_i > 0$, $i = 1, 2, ..., k$ and $\delta \geq 0$, this implies that $\beta_i > 0$, $\forall i$. Now, using (3.49) in (3.47), we obtain
\[
(x - \bar{x})^T \left[ \sum_{i=1}^k \bar{\lambda}_i (\nabla_y f_i(\bar{x}, \bar{y}) + \bar{w}_i - \bar{U}_i(\nabla_x g_i(\bar{x}, \bar{y}) - \bar{t}_i)) \right] \geq 0, \quad \forall x \in C_1. \tag{3.50}
\]
Let $x \in C_1$. Then $x + \bar{x} \in C_1$, as $C_1$ is a closed convex cone. On substituting $x + \bar{x}$ in place of $x$ in (3.50), we get
\[
x^T \sum_{i=1}^k \bar{\lambda}_i [(\nabla_y f_i(\bar{x}, \bar{y}) + \bar{w}_i) - \bar{U}_i(\nabla_x g_i(\bar{x}, \bar{y}) - \bar{t}_i)] \geq 0, \tag{3.51}
\]
which in turn implies that for all $x \in C_1$, we have
\[
\sum_{i=1}^k \bar{\lambda}_i [(\nabla_y f_i(\bar{x}, \bar{y}) + \bar{w}_i) - \bar{U}_i(\nabla_x g_i(\bar{x}, \bar{y}) - \bar{t}_i)] \in C_1^*. \tag{3.52}
\]
Also, by letting $x = 0$ and $x = 2\bar{x}$, simultaneously in (3.50), yields
\[
\bar{x}^T \sum_{i=1}^k \bar{\lambda}_i [(\nabla_y f_i(\bar{x}, \bar{y}) + \bar{w}_i) - \bar{U}_i(\nabla_x g_i(\bar{x}, \bar{y}) - \bar{t}_i)] = 0, \tag{3.53}
\]
Using $\bar{p}_i = 0$ in (3.35), we get, $\gamma = \delta \bar{y}$ and $\delta > 0$, we have
\[
\bar{y} = \frac{\gamma}{\delta} \in C_2.
Since $\beta > 0$ by (3.26) and the fact that $\gamma = \delta \bar{y}$, we get $\bar{y} \in N_{D_i}(\bar{z}_i), i = 1, 2, \ldots, k$.

This implies

$$\bar{y}^T \bar{z}_i = s(\bar{y}|D_i), i = 1, 2, \ldots, k. \quad (3.54)$$

By (3.27) and hypothesis (v), we have $\bar{y} \in N_{F_i}(\bar{r}_i), i = 1, 2, \ldots, k$. Hence,

$$\bar{y}^T \bar{r}_i = s(\bar{y}|F_i), i = 1, 2, \ldots, k. \quad (3.55)$$

Combining (3.32), (3.54)-(3.55) and given equation (3.1), reduce to

$$(f_i(\bar{x}, \bar{y}) + \bar{x}^T \bar{w}_i - s(\bar{y}|D_i))$$

$$- \bar{U}_i(g_i(\bar{x}, \bar{y}) - \bar{x}^T t_i - s(\bar{y}|F_i)) = 0, \quad i = 1, 2, \ldots, k. \quad (3.56)$$

Therefore, (3.52)-(3.53) and (3.56) shows that $(\bar{x}, \bar{y}, \bar{U}, \bar{w}, \bar{t}, \bar{\lambda}, \bar{q} = 0)$ is a feasible solution of $(MFD)_V$.

Under the assumptions Theorems 3.1 or 3.2, if $(\bar{x}, \bar{y}, \bar{U}, \bar{w}, \bar{t}, \bar{\lambda}, \bar{q} = 0)$ is not an efficient solution of $(MFD)_V$, then there exists other feasible solution $(u, v, V, w, t, \lambda, q)$, of $(MFD)_V$, such that $\bar{U} \leq V$.

Since $(\bar{x}, \bar{y}, \bar{U}, \bar{w}, \bar{t}, \bar{\lambda}, \bar{p} = 0)$ is a feasible solution of $(MFP)_U$, by Weak duality theorem, we have $U \leq V$, hence the contradiction implies that $(\bar{x}, \bar{y}, \bar{U}, \bar{w}, \bar{t}, \bar{\lambda}, \bar{q} = 0)$ is an efficient solution of $(MFD)_V$. Hence, the result.

**Theorem 3.4.** (Strong duality). Let $(\bar{x}, \bar{y}, \bar{U}, \bar{z}, \bar{r}, \bar{\lambda}, \bar{p} = 0)$ be efficient solution of $(MFP)_U$ and fix $\lambda = \bar{\lambda}$ in $(MFD)_V$. Suppose that

1. $\nabla_x H_i(\bar{x}, \bar{y}, 0) = \nabla_x G_i(\bar{x}, \bar{y}, 0) = 0, \nabla_q \Phi_i(\bar{x}, \bar{y}, 0) = \nabla_q \Psi_i(\bar{x}, \bar{y}, 0) = 0, \ H_i(\bar{x}, \bar{y}, 0) = G_i(\bar{x}, \bar{y}, 0) = 0, \ \forall i = 1, 2, \ldots, k$.

2. $\bar{U}_i > 0, \forall i = 1, 2, \ldots, k$.

3. $\nabla_{p_i} H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{U}_i \nabla_{p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i)$ is nonsingular $\forall i = 1, 2, \ldots, k$.

4. $\sum_{i=1}^{k} \lambda_i (\nabla_{yy} f_i(\bar{x}, \bar{y}) - \bar{U}_i \nabla_{yy} g_i(\bar{x}, \bar{y}))$ is positive definite and $p_i^T ((\nabla_x H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{U}_i \nabla_x G_i(\bar{x}, \bar{y}, \bar{p}_i)) - (\nabla_{p_i} H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{U}_i \nabla_{p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i))) \geq 0, \ \forall i = 1, 2, \ldots, k$.

5. or $\sum_{i=1}^{k} \lambda_i (\nabla_{yy} f_i(\bar{x}, \bar{y}) - \bar{U}_i \nabla_{yy} g_i(\bar{x}, \bar{y}))$ is negative definite and $p_i^T ((\nabla_x H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{U}_i \nabla_x G_i(\bar{x}, \bar{y}, \bar{p}_i)) - (\nabla_{p_i} H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{U}_i \nabla_{p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i))) \leq 0, \ \forall i = 1, 2, \ldots, k$.

6. the set of vectors $\{\nabla f_i(\bar{x}, \bar{y}, 0) - \bar{z}_i + \nabla p_i H_i(\bar{x}, \bar{y}, 0) - \bar{U}_i \nabla g_i(\bar{x}, \bar{y}) + \bar{r}_i + \nabla_{p_i} G_i(\bar{x}, \bar{y}, 0) : i = 1, 2, \ldots, k\}$ is linearly independent.

Then $\bar{p} = 0$, and there exists $\bar{w}_i \in Q_i, \bar{t}_i \in E_i, \ i = 1, 2, \ldots, k$ such that $(\bar{x}, \bar{y}, \bar{U}, \bar{w}, \bar{t}, \bar{\lambda}, \bar{q} = 0)$ is a feasible solution of $(MFD)_V$. Furthermore, if the hypotheses in theorem (3.1) or (3.2) are satisfied, then $(\bar{x}, \bar{y}, \bar{U}, \bar{w}, \bar{t}, \bar{\lambda}, \bar{q} = 0)$ is efficient solution of $(MFD)_V$, and the two objective values are equal.

**Proof.** It follows on the lines of Theorem 3.3. □

**Theorem 3.5.** (Strict converse duality). Let $(\bar{u}, \bar{v}, \bar{V}, \bar{w}, \bar{t}, \bar{\lambda}, \bar{q} = 0)$ be efficient solution of $(MFP)_U$ and fix $\lambda = \bar{\lambda}$ in $(MFD)_V$. If the following conditions hold

1. $\nabla_x \Phi_i(\bar{u}, \bar{v}, 0) = \nabla_x \Psi_i(\bar{u}, \bar{v}, 0) = 0, \nabla_q \Phi_i(\bar{u}, \bar{v}, 0) = \nabla_q \Psi_i(\bar{u}, \bar{v}, 0) = 0, \ H_i(\bar{u}, \bar{v}, 0) = G_i(\bar{u}, \bar{v}, 0) = 0, \ \forall i = 1, 2, \ldots, k$.

2. $\nabla_y \Phi_i(\bar{u}, \bar{v}, 0) = 0, \nabla_p H_i(\bar{u}, \bar{v}, 0) = \nabla_p G_i(\bar{u}, \bar{v}, 0) = 0, \ i = 1, 2, \ldots, k$,
Proof. Theorem 3.6. 

1. for all $i \in \{1, 2, \ldots, k\}$, the Hessian matrix $\nabla_{p,p} \Phi_i(\bar{u}, \bar{v}, \bar{q}_i) - \bar{V}_i \nabla_{p,p} \Psi_i(\bar{u}, \bar{v}, \bar{q}_i)$ is positive or negative definite.

2. the set of vectors $\{\nabla_{x,f_i}(\bar{u}, \bar{v}) + \bar{w}_i + \nabla_{q} \Phi_i(\bar{u}, \bar{v}, \bar{q}_i) - \bar{V}_i(\nabla_{x,g_i}(\bar{u}, \bar{v}) - \bar{t}_i + \nabla_{q} \Psi_i(\bar{u}, \bar{v}, \bar{q}_i))\}_{i=1}^k$ is linearly independent.

3. for $\bar{q}_i \in R^n$, $\bar{q}_i \not= 0$, $(i = 1, 2, \ldots, k)$ implies that

$$\sum_{i=1}^k \xi_i \bar{q}_i^T [\nabla_{x,f_i}(\bar{u}, \bar{v}) + \bar{w}_i + \nabla_{q} \Phi_i(\bar{u}, \bar{v}, \bar{q}_i) - \bar{V}_i(\nabla_{x,g_i}(\bar{u}, \bar{v}) - \bar{t}_i + \nabla_{q} \Psi_i(\bar{u}, \bar{v}, \bar{q}_i))] \not= 0,$$

4. $\bar{V}_i > 0$, $\forall i \in \{1, 2, \ldots, k\}$.

Then

(a) $\bar{q}_i = 0$, $i = 1, 2, \ldots, k$,

(b) there exists $\bar{z}_i \in D_i$ and $\bar{r}_i \in F_i$, $i = 1, 2, \ldots, k$ such that $(\bar{u}, \bar{v}, \bar{V}, \bar{z}, \bar{r}, \bar{\lambda}, \bar{\rho} = 0)$ is a feasible solution of $(MFD)_U$.

Furthermore, if the hypotheses in Theorem 3.1 or 3.2 are satisfied, then $(\bar{u}, \bar{v}, \bar{V}, \bar{z}, \bar{r}, \bar{\lambda}, \bar{\rho} = 0)$ is an efficient solution of $(MFD)_U$, and the two objective values are equal.

Proof. It follows on the lines of Theorem 3.3.

Theorem 3.6. (Strict converse duality). Let $(\bar{u}, \bar{v}, \bar{V}, \bar{\lambda}, \bar{\rho})$ be efficient solution of $(MFP)_V$ and fix $\lambda = \lambda$ in $(MFD)_V$. Suppose that

1. $\nabla_{x} \Phi_i(\bar{u}, \bar{v}, 0) = \nabla_{x} \Psi_i(\bar{u}, \bar{v}, 0) = 0$, $\nabla_{q} \Phi_i(\bar{u}, \bar{v}, 0) = \nabla_{q} \Psi_i(\bar{u}, \bar{v}, 0) = 0$,

2. $H_i(\bar{u}, \bar{v}, 0) = G_i(\bar{u}, \bar{v}, 0) = 0$, $\Phi_i(\bar{u}, \bar{v}, 0) = \Psi_i(\bar{u}, \bar{v}, 0) = 0$, $\nabla_{q} \Phi_i(\bar{u}, \bar{v}, 0) = \nabla_{q} \Psi_i(\bar{u}, \bar{v}, 0) = 0$, $i = 1, 2, \ldots, k$,

3. $\bar{V}_i > 0$, $\forall i \in \{1, 2, \ldots, k\}$,

4. $\nabla_{q} \Phi_i(\bar{u}, \bar{v}, \bar{q}_i) - \bar{V}_i \nabla_{q} \Psi_i(\bar{u}, \bar{v}, \bar{q}_i)$ is nonsingular $\forall i = 1, 2, \ldots, k$,

5. $\sum_{i=1}^k \lambda_i (\nabla_{x} f_i(\bar{u}, \bar{v}) - \bar{V}_i \nabla_{x} g_i(\bar{u}, \bar{v}))$ is positive definite and $\bar{q}_i^T (\nabla_{x} \Phi_i(\bar{u}, \bar{v}, \bar{q}_i) - \bar{V}_i \nabla_{x} \Psi_i(\bar{u}, \bar{v}, \bar{q}_i)) \geq 0$, $\forall i = 1, 2, \ldots, k$,

or

6. $\sum_{i=1}^k \lambda_i (\nabla_{x} f_i(\bar{u}, \bar{v}) - \bar{V}_i \nabla_{x} g_i(\bar{u}, \bar{v}))$ is negative definite and $\bar{q}_i^T (\nabla_{x} \Phi_i(\bar{u}, \bar{v}, \bar{q}_i) - \bar{V}_i \nabla_{x} \Psi_i(\bar{u}, \bar{v}, \bar{q}_i)) \leq 0$, $\forall i = 1, 2, \ldots, k$,

7. the set of vectors $\{\nabla_{x} f_i(\bar{u}, \bar{v}) + \bar{w}_i + \nabla_{q} \Phi_i(\bar{u}, \bar{v}, \bar{q}_i) - \bar{V}_i(\nabla_{x} g_i(\bar{x}, \bar{y}) - \bar{t}_i + \nabla_{q} \Psi_i(\bar{u}, \bar{v}, \bar{q}_i))\}$ is linearly independent.

Then, $\bar{q} = 0$ and there exists $\bar{z}_i \in D_i$ and $\bar{r}_i \in F_i$, $i = 1, 2, \ldots, k$ such that $(\bar{u}, \bar{v}, \bar{V}, \bar{z}, \bar{r}, \bar{\lambda}, \bar{\rho} = 0)$ is a feasible solution of $(MFD)_U$. Furthermore, if the hypothesis in Theorems 3.1 or 3.2 are satisfied, then $(\bar{u}, \bar{v}, \bar{V}, \bar{z}, \bar{r}, \bar{\lambda}, \bar{\rho} = 0)$ is an efficient solution of $(MFD)_U$, and the two objective values are equal.

Proof. It follows on the lines of Theorem 3.3.

4. Special cases

We consider some of the special cases of the problems studied in section 3. In all the cases, $C_1 = R^n_+$ and $C_2 = R^n_+$.

(i) then, our problems $(MFP)_U$ and $(MFD)_V$ reduce to the problems studied in Ying [12].

(ii) If $k = 1$, $g_1(x, y) = 1$, $H_1(x, y, p_1) = \frac{1}{2} p_1^T \nabla_{yy} f_1(x, y) p_1$, $\Phi_1(u, v, q_1) = \frac{1}{2} q_1^T \nabla_{xx} f_1(u, v) q_1$, $q_1(u, v) = 1$, $F_1 = \{0\}$, $E_1 = \{0\}$, then, $(MFP)_U$ and $(MFD)_V$ reduce to the problems considered by Hou and Yang [5].
(iii) If \( g_i(x, y) = 1 \), \( E_i = \{0\} \), \( F_i = \{0\} \) for all \( i \in \{1, 2, ..., k\} \), then, \((MFP)_U\) and \((MFD)_V\) becomes the problems considered by Chen [1].

(iv) If \( g_i(x, y) = 1 \), \( E_i = \{0\} \), \( F_i = \{0\} \), \( H_i(x, y, p_i) = \frac{1}{2} p_i^T \nabla_{yy} f_i(x, y) p_i \), \( \Phi_i(u, v, q_i) = \frac{1}{2} q_i^T \nabla_{xx} f_i(u, v) q_i \), for all \( i \in \{1, 2, ..., k\} \) in \((MFP)_U\) and \((MFD)_V\), then, the problems reduce to the problems considered by Yang et al.[11].

5. Acknowledgement

The authors are thankful to the reviewers for their valuable suggestions. The first author is thankful to the "Ministry of Human Resource and Development" India for financial support to carry out the above research work.

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