APPLICATIONS ON DIFFERENTIAL SUBORDINATION INVOLVING LINEAR OPERATOR

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**ABSTRACT.** In the present paper, we introduce and investigate some subclasses of strongly close-to-convex functions associated with the linear operator of meromorphic p-valently functions and study several inclusion relationships with some properties of this operator.

**KEYWORDS** linear operator, meromorphic functions, differential subordination, strongly close-to-convex functions, p-valently functions.

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1. **INTRODUCTION**

Let \( \sum_p \) denote the class of meromorphic functions of the form

\[
f(z) = z^{-p} + \sum_{k=0}^{\infty} a_{k+p} z^{k+p},
\]

(1.1)

which are analytic and \( p \)-valently in the punctured unit disk \( \mathcal{U}^* = \{ z : z \in \mathbb{C} : 0 < |z| < 1 \} = \mathcal{U} - 0 \).

If \( f(z) \) and \( g(z) \) are analytic in \( \mathcal{U} \), we say that \( f(z) \) is subordinate to \( g(z) \), written \( f \prec g \) or \( f(z) \prec g(z) \), if there exists a Schwarz function \( w(z) \), which (by definition) is analytic in \( \mathcal{U} \) such that \( f(z) = g(w(z)) \).

A function \( f(z) \in \sum_p \) is said to be \( p \)-valent meromorphic starlike of order \( \alpha (0 \leq \alpha \leq p) \) if it satisfies

\[
\text{Re}\left\{ -\frac{zf'(z)}{f(z)} \right\} > \alpha, \quad (z \in \mathcal{U})
\]

(1.2)

and the class of such functions is defined by \( MS^*(\alpha) \).

Furthermore, a function \( f(z) \in \sum_p \) is said to be \( p \)-valently meromorphic convex functions of order \( \alpha (0 \leq \alpha \leq p) \) if it satisfies

\[
\text{Re}\left\{ -(1 + \frac{zf''(z)}{f'(z)}) \right\} > \alpha, \quad (z \in \mathcal{U})
\]

(1.3)

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and the class of such functions is defined by $MK(\alpha)$.

Let $f(z) \in \sum_p$ and $g(z) \in MS^*(\alpha)$. Then $f(z) \in MC(\alpha, \beta)$ if and only if

$$Re\left\{ -\frac{zf'(z)}{g(z)} \right\} > \beta, \quad (z \in U),$$

(1.4)

where $0 \leq \alpha < p$ and $0 \leq \beta < p$. Such functions are called close-to-convex functions of order $\beta$ and type $\alpha$ in $U$, (see for details [4], [9]).

Further, a function $f(z) \in \sum_p$ is called $p$-valently meromorphic strongly starlike of order $\gamma (0 < \gamma \leq p)$ and type $\alpha (0 < \gamma \leq p)$ in $U$ if it satisfies

$$\left| \arg\left(-\frac{zf'(z)}{f(z)} - \alpha\right) \right| < \frac{\pi}{2\gamma}, \quad (z \in U),$$

(1.5)

and denoted by $MS^*(\gamma, \alpha)$.

If $f(z) \in \sum_p$ satisfies

$$\left| \arg\left(-(1 + \frac{zf'(z)}{f(z)}) - \alpha\right) \right| < \frac{\pi}{2\gamma}, \quad (z \in U),$$

for some $\gamma (0 < \gamma \leq p)$ and $\alpha (0 < \alpha \leq p)$, then $f$ is called $p$-valently meromorphic strongly convex of order $\gamma$ and type $\alpha$ in $U$ and denoted by $MC(\gamma, \alpha)$. We note that the classes mentioned above are the familiar classes which have been studied by many authors (see for example,[3],[6],[9],[10]).

For a function $f(z) \in \sum_p$ given by (1), we define a linear operator $D^n$ by

$$D^0 f(z) = f(z)$$

$$D^1 f(z) = z^{-p}(z^{p+1}f'(z))' = z^{-p} + \sum_{k=0}^{\infty} (2p + k + 1)a_{k+p}z^{k+p}$$

and

$$D^n f(z) = D(D^{n-1} f(z)) = z^{-p}(z^{p+1}D^{n-1}f(z))' = z^{-p} + \sum_{k=0}^{\infty} (2p + k + 1)^n a_{k+p}z^{k+p}. \quad (n \in N)$$

(1.6)

Using the relation (6), it is easy to verify that

$$z(D^n f(z))' = D^{n+1} f(z) - (p + 1)D^nf(z).$$

(1.7)

Also, we note that $D^n f(z)$ of another form of function studied by Liu and Srivastava [7], Srivastava and Patel [13] who introduce several inclusion relationships by using various subclasses of meromorphic $p$-valent function. A special cases of linear operator $D^n$ for $p = 1$ studied by Uralegaddi and Somanatha [14], Aouf and Hossen,[1], and got interesting results by using the operator $D^n$.

For $n \in N$, let $MC^{n+1}_p(\alpha, \beta, \gamma, A, B)$ be the class of functions $f(z) \sum_p$ satisfying the condition:

$$\left| -\arg\frac{z(D^{n+1}f(z))'}{D^{n+1}g(z)} - \gamma \right| < \frac{\pi}{2\delta} \quad (0 < \gamma \leq p, 0 \leq \delta < p; z \in U),$$

(1.8)

for some $g(z) \in S^{n+1}_p(\alpha, A, B)$, where

$$S^{n+1}_p(\alpha, A, B) = \left\{ g : \frac{1}{p + \alpha} \left( \frac{z(D^{n+1}g(z))'}{D^{n+1}g(z)} - \alpha \right) < \frac{1 + Az}{1 + Bz} \right\}$$

(1.9)

$$0 \leq \alpha < p, -1 \leq B \leq A \leq 1, z \in U \text{ and } g \in \sum_p$$

and the functions $f$ belonging to this class is called strongly close-to-convex function. In this study and by using the technique of Cho[2], we find some argument properties of functions belonging to $\sum_p$ which
include inclusion relationship and we obtain some interesting results for the functions class $MC_{p}^{n+1}(\alpha, \beta, \gamma, A, B)$ which we have defined here by the operator $D^n$.

To establish our main results, we shall need the following lemmas.

**Lemma 1.1** [5]: Let $h(z)$ be convex univalent in $\mathcal{U}$ with $h(0) = 1$ and $\Re\{zh(z) + \eta\} > 0(\varepsilon, \eta \in \mathbb{C})$. If $p(z)$ is analytic in $\mathcal{U}$ with $p(0) = 1$, then

$$p(z) + \frac{zp'(z)}{zp(\eta)} < h(z) \quad (z \in \mathcal{U}),$$

implies $p(z) < h(z)$ \quad (z \in \mathcal{U}).$

**Lemma 1.2** [8]: Let $h(z)$ be convex univalent in $\mathcal{U}$ and $w(z)$ be analytic in $\mathcal{U}$ with $\Re\{w(z)\} \geq 0$. If $p(z)$ is analytic in $\mathcal{U}$ with $p(0) = h(0)$, then

$$p(z) + w(z)p'(z) - \frac{p}{2} < h(z) \quad (z \in \mathcal{U}),$$

implies $p(z) < h(z)$ \quad (z \in \mathcal{U}).$

**Lemma 1.3** [9]: Let $p(z)$ be analytic in $\mathcal{U}$ with $p(0) = 1$ and $p(z) \neq 0$ in $\mathcal{U}$. If there exists two points $z_1, z_2$ in $\mathcal{U}$ such that

$$-\frac{\pi}{2} \xi_1 = \arg p(z_1) < \arg p(z) < \arg p(z_2) = \frac{\pi}{2} \xi_2 \quad (1.10)$$

for some $\xi_1, \xi_2 (\xi_1, \xi_2 > 0)$ and for all $z(|z| < |z_1| = |z_2|)$, then we have

$$\frac{z_1p'(z_1)}{p(z_1)} = i \frac{\xi_1 + \xi_2}{2} m \quad (1.11)$$

and

$$\frac{z_2p'(z_2)}{p(z_2)} = i \frac{\xi_1 + \xi_2}{2} m,$$

where $m \geq \frac{1-|z|}{1+|z|}$ and

$$c = i \tan \frac{\pi}{4} \left( \frac{\xi_2 - \xi_1}{\xi_1 + \xi_2} \right). \quad (1.12)$$

2. MAIN RESULTS

We first derive the following with use of Lemma 1.1.

**Proposition 2.1.** Let $h(z)$ be convex univalent in $\mathcal{U}$ with $h(0) = 1$ and $\Re\{h(z)\} > 0$.

If a function $f(z) \in \sum_p$ satisfies the following condition:

$$-\left( \frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)} - \alpha \right) < h(z),$$

then

$$-\left( \frac{z(D^n f(z))'}{D^n f(z)} - \alpha \right) < h(z), \quad (0 \leq \alpha < p; z \in \mathcal{U})$$

**Proof.** Let

$$p(z) = -\frac{1}{p + \alpha} \left( \frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)} - \alpha \right). \quad (2.1)$$

Then $p(z)$ is analytic function in $\mathcal{U}$ with $p(0) = 1$. By using (1.7), we obtain

$$p + 1 + \alpha + (p + \alpha)p(z) = -\frac{D^{n+1} f(z)}{D^n f(z)}. \quad (2.2)$$
Differentiating Logarithmically with respect to $z$ and multiplying by $z$, we get

$$p(z) + \frac{zp'(z)}{p + 1 + \alpha + (p + \alpha)p(z)} = -\frac{1}{p + \alpha} \left( \frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)} - \alpha \right).$$

Now, by using Lemma 1.1, we obtain

$$-\frac{1}{p + \alpha} \left( \frac{z(D^{n}f(z))'}{D^{n}f(z)} - \alpha \right) < h(z),$$
deduce that $p(z) < h(z)$.

Setting $h(z) = \frac{1 + A\zeta}{1 + B\zeta}$ $(-1 \leq B \leq A \leq 1)$, in Lemma 2.1, we obtain

**Corollary 2.1:** For $n \in N$ and $p \in \{1, 2, \ldots\}$, we have

$$S^{n+1}_{p}(\alpha, A, B) \subset S^{n}_{p}(\alpha, A, B).$$

**Proposition 2.2:** Let $h(z)$ be convex univalent in $\mathcal{U}$ with $h(0) = 1$ and $Re\{h(z)\} > 0$. If $f(z) \in \sum_{p}$ satisfies

$$\frac{1}{p + \alpha} \left( \frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)} - \alpha \right) < h(z),$$
then

$$-\frac{1}{p + \alpha} \left( \frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)} - \alpha \right) < h(z),$$

$$0 \leq \alpha < p; z \in \mathcal{U}$$

where

$$I_{\theta} f(z) = \frac{\theta - p}{z^p} \int_{0}^{2\pi} e^{\theta-1} f(t) dt \quad (\theta \geq 0) \quad (2.3)$$

**Proof.** From (2.3), we have

$$z(D^{n+1}I_{\theta} f(z))' = (\theta - p)(D^{n+1}f(z)) - \theta(D^{n+1}f(z)). \quad (2.4)$$

Let

$$p(z) = -\frac{1}{p + \alpha} \left( \frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)} - \alpha \right),$$
p($z$) is analytic function in $\mathcal{U}$ with $p(0) = 1$. Then from (2.4), we get

$$\theta + \alpha + (p + \alpha)p(z) = -\theta - p + \frac{D^{n+1}f(z)}{D^{n+1}I_{\theta} f(z)}. \quad (2.5)$$

By differentiating (2.5) logarithmically with respect to $z$ and multiplying by $z$, we have

$$p(z) + \frac{zp'(z)}{\theta + \alpha + (p + \alpha)p(z)} = -\frac{1}{p + \alpha} \left( \frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)} - \alpha \right).$$

Thus, by Lemma 1.1, we get

$$-\frac{1}{p + \alpha} \left( \frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)} - \alpha \right) < h(z).$$

Taking $h(z) = \frac{1 + A\zeta}{1 + B\zeta}$ $(-1 \leq B \leq A \leq 1)$, in Proposition 2.2, we obtain

**Corollary 2.2:** If $f(z) \in S^{n+1}_{p}(\alpha, A, B)$, then $I_{\theta} f(z) \in S^{n+1}_{p}(\alpha, A, B)$. Hence on Applying Proposition 2.2, we prove the following theorem

**Theorem 2.1:** Let $f(z) \in \sum_{p}$ and $(0 < \delta, \delta_{2} \leq p, 0 \leq \alpha < p)$. If

$$\frac{\pi}{2} \delta < \arg\left( -\frac{z(D^{n+1}f(z))'}{D^{n+1}g(z)} - \gamma \right) < \frac{\pi}{2} \delta_{2}$$

for some $g(z) \in S^{n+1}_{p}(\alpha, A, B)$, then
Therefore, by (2.10) and (2.11), we obtain

$$\frac{\pi}{2} \beta_1 < \arg\left(-\frac{z(D^n f(z))'}{D^n g(z)} - \gamma\right) < \frac{\pi}{2} \beta_2,$$

where $\beta_1$ and $\beta_2 (0 < \beta_1, \beta_2 \leq p)$ are the solution of the equations:

$$\delta_1 = \begin{cases} 
\beta_1 + \frac{2}{\pi} \tan^{-1} \left\{ \frac{(\beta_1 + \beta_2)(1 - |c|) \cos \frac{\pi}{2} t_1}{z^2(1 - p + \alpha) + p + 1 + \alpha(1 - |c|) + (\beta_1 + \beta_2)(1 - |c|) \sin \frac{\pi}{2} t_1} \right\} & B \neq -1 \\
\beta_1 & \text{otherwise}
\end{cases},$$

(2.6)

and

$$\delta_2 = \begin{cases} 
\beta_2 + \frac{2}{\pi} \tan^{-1} \left\{ \frac{(\beta_1 + \beta_2)(1 - |c|) \cos \frac{\pi}{2} t_1}{z^2(1 - p + \alpha) + p + 1 + \alpha(1 - |c|) + (\beta_1 + \beta_2)(1 - |c|) \sin \frac{\pi}{2} t_1} \right\} & B \neq -1 \\
\beta_2 & \text{otherwise}
\end{cases},$$

(2.7)

where $c$ is given by (1.12) and $t_1 = \frac{2}{\pi} \sin^{-1} \left( \frac{(p + \gamma)(1 - B)}{(p + \alpha)(1 - A) + p + 1 + \alpha(1 - |c|)} \right)$.

**Proof.** Let

$$p(z) = -\frac{1}{p + \alpha} \left( \frac{z(D^n f(z))'}{D^n g(z)} - \gamma \right).$$

(2.8)

It follows from (1.7) that

$$[(p + \gamma)(p(z) - \gamma)] D^n g(z) = D^{n+1} f(z) - (p + 1) D^n f(z).$$

(2.9)

Differentiating both sides of (2.9), and multiplying by $z$, we deduce that

$$(p + \gamma) z p'(z) D^n g(z) + [(p + \gamma) p(z) - \gamma] z (D^n g(z))' = z (D^{n+1} f(z))' - (p + 1) z (D^n f(z))'.$$

(2.10)

Since $g(z) \in S_p^{n+1}(\alpha, A, B)$, by applying Corollary 2.1, we find that $g(z) \in S_p^n(\alpha, A, B)$. Thus, by using (1.7) and put $q(z) = -\frac{1}{p + \alpha} \left( \frac{z(D^n g(z))'}{D^n g(z)} - \alpha \right)$, we immediately have

$$(p + \alpha) q(z) + \alpha + p + 1 = -\frac{D^{n+1} g(z)}{D^n g(z)}.$$  

(2.11)

Therefore, by (2.10) and (2.11), we obtain

$$-\frac{1}{p + \alpha} \left( \frac{z(D^{n+1} f(z))'}{D^{n+1} g(z)} - \gamma \right) = p(z) + \frac{zp'(z)}{(p + \alpha) q(z) + \alpha + p + 1}.$$  

Making use the result of Silverman and Silvia [10], we obtain

$$|q(z) - \frac{1 - AB}{1 - B^2}| < \frac{A - B}{1 - B^2} \quad (z \in \mathcal{U}; B \neq -1)$$

(2.12)

and

$$\text{Re} \{q(z)\} > \frac{1 - A}{2} \quad (z \in \mathcal{U}; B = -1)$$

(2.13)

It follows from (2.12) and (2.13) that

$$(p + \alpha) q(z) + p + \alpha + 1 = re^{i\frac{\pi}{2}}.$$  

Now, if $B \neq -1$, we have

$$\frac{(p + \alpha)(1 - A)}{1 - B} + \alpha + p + 1 < r < \frac{(p + \alpha)(1 + A)}{1 + B} + \alpha + p + 1, \quad -t_1 < \phi < t_1,$$

and if $B = -1$, we have

$$\frac{(p + \alpha)(1 - A)}{2} + \alpha + p + 1 < r < \infty, \quad -1 < \phi < 1.$$

Applying Lemma 1.2 with $w = -\frac{1}{(p + \alpha) q(z) + p + \alpha + 1}$, we note that $p(z)$ is analytic with $p(0) = 1$ and $\text{Re} \{p(z)\} > 0$ in $\mathcal{U}$. 
Hence by Lemma 1.3 for \( z_1, z_2 \in \mathbb{U} \), such that the condition (1.10) is satisfied, then we obtain (1.11) under the restriction (1.12). On other hand, if \( B \neq -1 \), we readily get

\[
\arg(-p(z_1) + \frac{z_1 p'(z_1)}{(p+\alpha)q(z_1) + p + \alpha + 1}) = -\frac{\pi}{2} \beta_1 + \arg(1 - \frac{\beta_1 + \beta_2}{2} m(\rho e^{i\frac{\pi}{2}})^{-1})
\leq -\frac{\pi}{2} \beta_1 - \tan^{-1}\left(\frac{(\beta_1 + \beta_2)m \sin \frac{\pi}{2}(1 - \phi)}{2r + (\beta_1 + \beta_2)m \cos \frac{\pi}{2}(1 - \phi)}\right)
\]

\[
-\frac{\pi}{2} \beta_1 - \tan^{-1}\left(\frac{(\beta_1 + \beta_2)(1 - |c|) \cos \frac{\pi}{2} t_1}{2\frac{(p+\alpha)(1+\alpha)}{1+B} + p + \alpha + 1}(1 + |c|) + (\beta_1 + \beta_2)(1 - |c|) \sin \frac{\pi}{2} t_1\right) = -\frac{\pi}{2} \delta_1,
\]

and

\[
\arg(-p(z_2) + \frac{z_2 p'(z_2)}{(p+\alpha)q(z_2) + p + \alpha + 1}) \geq -\frac{\pi}{2} \beta_2 - \tan^{-1}\left(\frac{(\beta_1 + \beta_2)(1 - |c|) \cos \frac{\pi}{2} t_1}{2\frac{(p+\alpha)(1+\alpha)}{1+B} + p + \alpha + 1}(1 + |c|) + (\beta_1 + \beta_2)(1 - |c|) \sin \frac{\pi}{2} t_1\right) = -\frac{\pi}{2} \delta_2.
\]

Also, if \( B = -1 \), we readily get

\[
\arg(-p(z_1) + \frac{z_1 p'(z_1)}{(p+\alpha)q(z_1) + p + \alpha + 1}) \leq -\frac{\pi}{2} \beta_1
\]

and

\[
\arg(-p(z_2) + \frac{z_2 p'(z_2)}{(p+\alpha)q(z_2) + p + \alpha + 1}) \geq -\frac{\pi}{2} \beta_2
\]

There are contradiction with a assumption. This completes the proof of Theorem 2.1

**Corollary 2.3:**

\[
MC^{n+1}_p(\alpha, \beta, \gamma, A, B) \subset MC^n_p(\alpha, \beta, \gamma, A, B).
\]

Setting \( n = 0, \delta_1 = \delta_2 = \delta \) in Theorem 2.1, we get:

**Corollary 2.4:** Let \( f(z) \in \sum_p \). If

\[
\left|\frac{z(z^{-p}(z^{p+1}f)')'}{z^{-p}(z^{p+1}g(z))'} - \gamma\right| < \frac{\pi}{2} \delta
\]

for some \( g(z) \in S_p^1 \), then where \( \beta(0 < \beta \leq p) \) is the solution of equation:

\[
\delta = \left\{ \begin{array}{ll}
\beta + \frac{2}{\pi} \tan^{-1}\left\{\frac{\beta \cos \frac{\pi}{2} t_1}{\beta \cos \frac{\pi}{2} t_1 + p + \alpha + \beta \sin \frac{\pi}{2} t_1}\right\} & B \neq -1 \\
\beta & B = -1,
\end{array} \right.
\]

and \( t_1 = \frac{2}{\pi} \sin^{-1}\left(\frac{\beta(1-B)}{(p+\alpha)(1-AB) + (p+\alpha)(1-B)}\right) \).

**Theorem 2.2:** Let \( f(z) \in \sum_p \) and \( (0 < \delta_1, \delta_2 \leq 1, 0 \leq \gamma < 1) \). If

\[
-\frac{\pi}{2} \delta_1 \leq \arg(-\frac{z(D^{n+1}f(z))'}{D^n g(z)} - \gamma) \leq -\frac{\pi}{2} \delta_2,
\]

for some \( g(z) \in S_p^{n+1}(\alpha, A, B) \), then

\[
-\frac{\pi}{2} \beta_1 \leq \arg(-\frac{z(D^{n+1}g(z))'}{D^n g(z)} - \gamma) \leq -\frac{\pi}{2} \beta_2,
\]

where \( \|g\| \) is defined by (2.3), and \( \beta_1, \beta_2 \), are the solutions of

\[
\delta_1 = \left\{ \begin{array}{ll}
\beta_1 + \frac{2}{\pi} \tan^{-1}\left\{\frac{\beta_1 + \beta_2}{2\beta_1 + p+\alpha + \beta_2 \sin \frac{\pi}{2} t_2}\right\} & B \neq -1 \\
\beta_1 & B = -1,
\end{array} \right. \quad (2.14)
\]

and
\[
\delta_2 = \begin{cases}
\beta_2 + \frac{2}{\pi} \tan^{-1} \left\{ \frac{(\beta_1 + \beta_2)(1 - |c|) \cos \frac{2}{\pi} t_2 - 1}{(\beta_1 + \beta_2)(1 - |c|) \sin \frac{2}{\pi} t_2} \right\} & B \neq -1 \\
\beta_2 & B = -1,
\end{cases}
\quad (2.15)
\]

Here \( c \) is given by (1.12) and \( t_2 = \frac{2}{\pi} \sin^{-1} \left( \frac{(p + \alpha)(1 - B)}{(p + \alpha)(1 - AB) + (\theta + \alpha)(1 - B^2)} \right) \).

**Proof.** Let

\[
p(z) = -\frac{1}{p + \alpha} \left( \frac{z(D^{n+1}I_\theta f(z))'}{D^{n+1}I_\theta g(z)} - \gamma \right).
\]

Since \( g(z) \in S^{n+1}_p(\alpha, A, B) \), and by using Corollary 2.2, we obtain \( I_\theta g(z) \in S^{n+1}_p(\alpha, \beta, \gamma, A, B) \).

By using (2.5), we get

\[
[(p + \gamma)(p(z) - \gamma)]D^{n+1}I_\theta g(z) = (\theta - p)(D^{n+1}f(z)) - \theta D^{n+1}I_\theta f(z)
\]

and simplifying, we obtain

\[
(p + \gamma)z p'(z) + [(p + \gamma)p(z) + \gamma][(p + \alpha)q(z) + \theta + \alpha] = (\theta - p) \frac{z(D^{n+1}f(z))'}{D^{n+1}I_\theta g(z)},
\]

where

\[
g(z) = -\frac{1}{p + \alpha} \left( \frac{z(D^{n+1}I_\theta g(z))'}{D^{n+1}I_\theta g(z)} - \alpha \right).
\]

Therefore,

\[
-\frac{1}{p + \alpha} \left( \frac{z(D^{n+1}f(z))'}{D^{n+1}I_\theta g(z)} - \alpha \right) = p(z) + \frac{z p'(z)}{(p + \alpha)q(z) + \alpha + \theta}.
\]

Applying a similar method as in the proof of Theorem 2.1 we get the required result and the proof is complete.

Setting \( \delta_1 = \delta_2 = \delta \) in Theorem 2.2, we obtain

**Corollary 2.5:** Let \( f(z) \in \sum_p \) and \( 0 \leq \gamma < p, 0 < \delta \leq 1 \). If

\[
\left| \arg\left( -\frac{z(D^{n+1}f(z))'}{D^{n+1}I_\theta g(z)} - \gamma \right) \right| < \frac{\pi}{2} \delta
\]

for some \( g(z) \in S^{n+1}_p(\alpha, A, B) \), then

\[
\left| \arg\left( -\frac{z(D^{n+1}I_\theta g(f(z))'}{D^{n+1}I_\theta g(z)} - \gamma \right) \right| < \frac{\pi}{2} \beta,
\]

where \( \beta \) is given by (2.5), and \( \beta(0 < \beta \leq 1) \) is the solution of the equation

\[
\delta = \begin{cases}
\beta + \frac{2}{\pi} \tan^{-1} \left\{ \frac{\beta \cos \frac{2}{\pi} t_2}{(\beta + \alpha)(1 + |\alpha|) + \beta \sin \frac{2}{\pi} t_2} \right\} & B \neq -1 \\
\beta & B = -1
\end{cases}
\]

**Corollary 2.6:** If \( f(z) \in MC^{n+1}_p(\gamma, \delta, \alpha, A, B) \), then \( I_\theta f(z) \in MC^{n+1}_p(\gamma, \delta, \alpha, A, B) \).

**References**