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EXISTENCE OF SYSTEMS OF GENERALIZED VECTOR QUASI-EQUILIBRIUM PROBLEMS IN PRODUCT FC-SPACES[◇]

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ABSTRACT. In this paper, three classes of systems of generalized vector quasi-equilibrium problems are introduced and studied in product FC-spaces without convexity structure. We prove some new equilibrium existence theorems for three classes of systems of generalized vector quasi-equilibrium problems in noncompact product FC-spaces. These results improve and generalize some recent results in literature to product FC-spaces without any convexity structure.

KEYWORDS : Systems of generalized vector quasi-equilibrium problem; $C_i(x) - FC-$ diagonal quasi-convex; $C_i(x) - FC-$ quasi-convex; $C_i(x) - FC-$ quasi-convex-like; FC-spaces.

1. INTRODUCTION

Let X be a convex subset of a real topological vector space E (in short t.v.s.) and $F : X \times X \to R$ be a given function with $F(x, x) \geq 0$ for all $x \in X$. By equilibrium problem, Blum and Oettli [1] considered the problem of finding $u \in X$ such that $F(u, y) \geq 0$ for all $y \in X$. This problem contains optimization problems, Nash type equilibria problems, variational inequality problems, complementary problems and fixed point problems as special case. In 1980, Giannessi [2] introduced the vector variational inequality problem in finite dimensional Euclidean spaces. From the above applications, generalized vector quasi-equilibrium problems, and system of generalized vector quasi-equilibrium problems have become important developed directions of vector variational inequality theory, for example, see [4-26].

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In 2000, Ansari et al. [4] introduced the system of vector equilibrium problems (for short, SVEP), that is, a family of equilibrium problems for vector-valued bifunctions defined on a product set, with applications in vector optimization problems and Nash equilibrium problem [5] for vector valued functions. Recently, Ansari et al. [6] introduced the following concept of system of vector quasi-equilibrium problems (in short, SVQEP) as follows (see also in [7-13]; Let I be any index set and for each $i \in I$, let X_i be a topological vector space. Consider a family of nonempty convex subsets $\{K_i\}_{i\in I}$ with $K_i \subset X_i$. We denote by $K = \prod_{i\in I} K_i$ and $X = \prod_{i\in I} X_i$. For each $i \in I$, let Y_i be a topological vector space and let $C_i : K \longrightarrow 2^{Y_i}$, $S_i : K \longrightarrow 2^{K_i}$ and $F_i : K \times K_i \longrightarrow 2^{Y_i}$ be multi-valued mappings. The system of vector quasi-equilibrium problems (in short, SVQEP), that is, to find $x \in K$ such that for each $i \in I$,

$$x_i \in S_i(x) \ : \ F_i(x, y_i) \notin -intC_i(x) \ \forall y_i \in S_i(x). \tag{1.1}$$

If $S_i(x) = K_i$ for all $x \in K$, then (SVQEP) reduces to (SVEP) (see [4]) and if the index set I is singleton, then (SVQEP) becomes the vector quasi-equilibrium problem which contains vector quasi-optimization problem and vector quasi-variational inequality problem as special cases (see [3]).

In 2010, Li and Li [14] considered three following problems. Let let X, Y and Z be three real topological spaces, let X and Y be Hausdorff spaces, $E \subset X$ and $D \subset Z$ be two nonempty subsets. Let $C : X \to 2^Y$ be set-valued mapping such that $C(x)$ be a proper, closed and convex cone of Y with nonempty interior. Let $S : E \to 2^E, T : E \to 2^D$ and $F : E \times D \times E \to 2^Y$ be three set-valued maps. Three classes of generalized vector quasi-equlilbrium problems: Find $\bar{x} \in E$ and $\bar{z} \in T(\bar{x})$ such that

(i) (GVQEP I) $\bar{x} \in S(\bar{x})$ and $F(\bar{x}, \bar{z}) \nsubseteq -intC(\bar{x}), \forall y \in S(\bar{x})$.

(ii) (GVQEP II) $\bar{x} \in S(\bar{x})$ and $F(\bar{x}, \bar{z}) \cap -intC(\bar{x}) = \emptyset, \forall y \in S(\bar{x})$.

(iii) (GVQEP III) $\bar{x} \in S(\bar{x})$ and $F(\bar{x}, \bar{z}) \subset -C(\bar{x}), \forall y \in S(\bar{x})$.

Moreover, they obtained some existence results by using the well know Fan-KKM theorem without the compact assumption and unless otherwise specified.

On the other hand, it is well known that many existence theorems of maximal elements for set-valued mappings have been established in topological vector spaces, H-spaces and G-convex spaces by many authors. The notion of generalized convex (in short, G-convex) spaces was introduced by Park and Kim in [15, 16] . In 2005, Ding [17] was introduced the notion of a finitely continuous topological space (in short, FC-space). It is clear that the class of G-convex spaces is a subclass of FC-spaces. We emphasis that FC-space is a topological space without any convexity structure.

Motivated and inspired by research works mentioned above, in this paper, we introduce three classes of systems of generalized vector quasi-equilibrium problems in product FC-spaces. Let X and Y be two nonempty sets. We denote by 2^Y and $\langle X \rangle$ the family of all subsets of Y and the family of all nonempty finite subsets of X, respectively. Let I be any index set. For each $i \in I$, let X_i and Y_i be topological spaces and Z_i be a topological vector space. Let $X = \prod_{i\in I} X_i, Y = \prod_{i\in I} Y_i$ and for $i \in I$ and $z \in X, z_i = \pi_i(z)$ be the projection of z onto X_i. For each $i \in I$, let $A_i : Y \times X \to 2^{X_i}, T_i : Y \times X \to 2^{Y_i}, C_i : X \to 2^{Z_i}$ such that for each $z \in X, C_i(z)$ be a closed convex cone with nonempty interior, and $\Psi_i : X_i \times Y \times X \to 2^{Z_i}$ be set-valued mappings.

We consider the following three classes of systems of generalized vector quasi-equilibrium problems:

(i) (SGVQEP 1) Find $(\hat{y}, \hat{z}) \in Y \times X$ such that for each $i \in I$,

$$\hat{y}_i \in T_i(\hat{y}, \hat{z}), \hat{z}_i \in A_i(\hat{y}, \hat{z}) \; and \; \Psi_i(x_i, \hat{y}, \hat{z}) \nsubseteq -intC_i(\hat{z}), \forall x_i \in A_i(\hat{y}, \hat{z}).$$

(ii) (SGVQEP 2) Find $(\hat{y}, \hat{z}) \in Y \times X$ such that for each $i \in I$,

$$\hat{y}_i \in T_i(\hat{y}, \hat{z}), \hat{z}_i \in A_i(\hat{y}, \hat{z}) \; and \; \Psi_i(x_i, \hat{y}, \hat{z}) \cap -intC_i(\hat{z}) = \emptyset, \forall x_i \in A_i(\hat{y}, \hat{z}).$$

(iii) (SGVQEP 3) Find $(\hat{y}, \hat{z}) \in Y \times X$ such that for each $i \in I$,

$$\hat{y}_i \in T_i(\hat{y}, \hat{z}), \hat{z}_i \in A_i(\hat{y}, \hat{z}) \; and \; \Psi_i(x_i, \hat{y}, \hat{z}) \subset -C_i(\hat{z}), \forall x_i \in A_i(\hat{y}, \hat{z}).$$

Let V_0 be a topological vector space ordered by a proper closed convex cone D in V_0 and let $h : Y \times X \to 2^{V_0}$ is a set-valued mapping. Moreover, we introduce the notations of $C_i(z) - FC-$diagonal quasi-convex, $C_i(x) - FC-$quasi-convex and $C_i(x) - FC-$quasiconvex-like for set-valued mappings in FC-space. By using these notions and an existence theorem of maximal elements for a family of set-valued mappings, we prove some new existence theorems of solutions for the SGVQEP (1), SGVQEP (2) and SGVQEP (3) in noncompact product FC-spaces without convexity structure. These results improve and generalize some recent known results in literature to noncompact FC-spaces.

2. PRELIMINARIES

Let Δ_n be the standard n-dimensional simplex with vertices $\{e_0, e_1, \dots, e_n\}$. If J is a nonempty subset of $\{0, 1, \dots, n\}$, we denote by Δ_J the convex hull of the vertices $\{e_j : j \in J\}$. The following notion was introduced by Ben-El-Mechaiekh et al. [18].

Definition 2.1. (X, Γ) is said to be a L-convex space if X is a topological space and $\Gamma : \langle X \rangle \to 2^X$ is a mapping such that for each $N \in \langle X \rangle$ with $|N| = n+1$, there exists a continuous mapping $\varphi_N : \Delta_n \to \Gamma(N)$ satisfying $A \in \langle N \rangle$ with $|A| = J+1$ implies $\varphi_N(\Delta_J) \subset \Gamma(A)$, where Δ_J is the face of Δ_N corresponding to A.

The following notion of a finitely continuous topological space (in short, FC-space) was introduced by Ding [17].

Definition 2.2. (X, φ_N) is said to be a FC-space if X is a topological space and for each $N = \{x_0, \dots, x_n\} \in \langle X \rangle$ where some elements in N may be same, there exists a continuous mapping $\varphi_N : \Delta_n \to X$. A subset D of (X, φ_N) is said to be a FC-subspace of X if for each $N = \{x_0, \dots, x_n\} \in \langle X \rangle$ and for each $\{x_{i_0}, \dots, x_{i_k}\} \subset N \cap D, \varphi_N(\Delta_k) \subset D$, where $\Delta_k = co(\{e_{i_j} : j = 0, \dots, k\})$.

It is clear that any convex subset of a topological vector space, any H-space introduced by Horvath [19], any G-convex space introduced by Park and Kim [15, 16], and any L-convex spaces introduced by Ben-El-Mechaiekh et al. [18] are all FC-space.

By the definition of FC-subspaces of a FC-space, it is easy to see that if $\{B_i\} \in I$ is a family of FC-subspaces of a FC-space (Y, φ_N) and $\cap_{i \in I} B_i \neq \emptyset$, then $\cap_{i \in I} B_i$ is also a FC-subspace of (Y, φ_N) where I is any index set. For a subset A of (Y, φ_N), we can define the FC-hull of A as follows:

$$FC(A) = \bigcap \{B \subset Y : A \subset B \; and \; B \; is \; FC - subspace \; of \; Y\}.$$

Clearly, $FC(A)$ is the smallest FC-subspace of Y containing A and each FC-subspace of a FC-space is also a FC-space.

Lemma 2.3. [20] *Let (Y, φ_N) be a FC-space and A be a nonempty subset of Y. Then*

$$FC(A) = \bigcup\{FC(N) : N \in \langle A\rangle\}.$$

Lemma 2.4. [20] *Let X be a topological space, (Y, φ_N) be a FC-space and $G : X \to 2^Y$ be such that $G^{-1}(y) = \{x \in X : y \in G(x)\}$ is compactly open in X for each $y \in Y$. Then the mapping $FC(G) : X \to 2^Y$ defined by $FC(G)(x) = FC(G(x))$ for each $x \in X$ satisfies that $(FC(G))^{-1}(y)$ is also compactly open in X for each $y \in Y$.*

Lemma 2.5. [17] *Let I be any index set. For each $i \in I$, let (Y_i, φ_{N_i}) be a FC-space. Let $Y = \prod_{i\in I} Y_i$ and $\varphi_N = \prod_{i\in I} \varphi_{N_i}$. Then (Y, φ_N) is also a FC-space.*

Lemma 2.6. [20] *Let I be any index set. For each $i \in I$, let (X_i, φ_{N_i}) be a FC-space, $X = \prod_{i\in I} X_i$ and K be a compact subset of X. For each $i \in I$, let $G_i : X \to 2^{X_i}$ be such that*

- (i) *for each $i \in I$ and $x \in X, G_i(x)$ is a FC-subspace of X_i,*
- (ii) *for each $x \in X$, $\pi_i(x) \notin G_i(x)$ for all $i \in I$,*
- (iii) *for each $y_i \in X_i, G_i^{-1}(y_i)$ is compactly open in X*
- (iv) *for each $N_i \in \langle X_i\rangle$, there exists a nonempty compact FC-subspace L_{N_i} of X_i containing N_i and for each $x \in X \setminus K$, there exists $i \in I$ satisfying $L_{N_i} \cap G_i(x) \neq \emptyset$.*

Then there exists $\hat{x} \in K$ such that $G_i(\hat{x}) = \emptyset$, for each $i \in I$.

Lemma 2.7. [21] *Let X and Y be topological spaces and $G : X \to 2^Y$ be a set-valued mapping. Then G is lower semicontinuous in $x \in X$ if and only if for any $y \in G(x)$ and any net $\{x_\alpha\} \subset X$ satisfying $x_\alpha \to x$, there exists a net $\{y_\alpha\}$ such that $y_\alpha \in G(x_\alpha)$ and $y_\alpha \to y$.*

Lemma 2.8. [22] *Let X, Y and Z be topological spaces. Let $F : X \times Y \to 2^Z$ and $C : X \to 2^Z$ be set-valued mappings such that*

- (i) *C has closed (resp., open) graph,*
- (ii) *for each $y \in Y, F(\cdot, y)$ is lower semicontinuous on each compact subset of X.*

Then the mapping $F^* : Y \to 2^X$ defined by $F^*(y) = \{x \in X : F(x, y) \subset C(x)\}$ (resp., $F^*(y) = \{x \in X : F(x, y) \cap C(x) = \emptyset\}$) has compactly closed values.

Lemma 2.9. [22] *Let X, Y and Z be topological spaces. Let $F : X \times Y \to 2^Z$ and $C : X \to 2^Z$ be set-valued mappings such that*

- (i) *C has closed (resp., open) graph in $X \times Z$,*
- (ii) *for each $y \in Y, F(\cdot, y)$ is upper semicontinuous on each compact subset of X with nonempty compactly closed values.*

Then the mapping $F^* : Y \to 2^X$ defined by $F^*(y) = \{x \in X : F(x, y) \not\subseteq C(x)\}$ (resp., $F^*(y) = \{x \in X : F(x, y) \cap C(x) \neq \emptyset\}$) has compactly closed values.

3. THE EXISTENCE OF THE SYSTEM OF GENERALIZED VARIATIONAL INEQUALITY

Throughout this section, unless otherwise specified, we assume the following notations and assumptions. Let I be any index set. For each $i \in I$, let (X_i, φ_{N_i}) and (Y_i, φ'_{N_i}) be $FC-$spaces, and Z_i be a nonempty set. Let $X = \prod_{i\in I} X_i$ and $Y = \prod_{i\in I} Y_i$. For each $i \in I$, let $A_i : Y \times X \to 2^{X_i}, T_i : Y \times X \to 2^{Y_i}$and $C_i : Y \times X \to 2^{Z_i}$ such that for each $z \in X, C_i(z)$ be a closed convex cone with nonempty interior, $\Psi_i : X_i \times Y \times X \to 2^{Z_i}$ be set valued mappings. From Li and Li [14] and Ding [22], we first propose the following generalized convexity definitions.

Definition 3.1. Let I be any index set. For each $i \in I$, let (X_i, φ_{N_i}) and (Y_i, φ'_{N_i}) be $FC-$spaces, and Z_i be a nonempty set. Let $X = \prod_{i\in I} X_i$ and $Y = \prod_{i\in I} Y_i$. For each $i \in I$, let $A_i : Y \times X \to 2^{X_i}, T_i : Y \times X \to 2^{Y_i}$and $C_i : Y \times X \to 2^{Z_i}$ such that for each $z \in X, C_i(z)$ be a closed convex cone with nonempty interior, $\Psi_i : X_i \times Y \times X \to 2^{Z_i}$ be set valued mappings. For each $i \in I$ and $y \in Y, \Psi_i : X_i \times Y \times X \to 2^{Z_i}$ is said to be

(i) Ψ_i is said to be $C_i(z) - FC-$ diagonal quasi-convex of weak type 1 in first argument if each $N_i = \{x_{i,0}, \dots, x_{i,n}\} \in \langle X_i\rangle$ and for each $z \in X$ with $z_i \in FC(N_i)$, there exists $j \in \{0, \dots, k\}$ such that

$$\Psi_i(x_{i_j}, y, z) \nsubseteq -intC_i(z),$$

(ii) Ψ_i is said to be $C_i(z) - FC-$ diagonal quasi-convex of weak type 2 in first argument if each $N_i = \{x_{i,0}, \dots, x_{i,n}\} \in \langle X_i\rangle$ and for each $z \in X$ with $z_i \in FC(N_i)$, there exists $j \in \{0, \dots, k\}$ such that

$$\Psi_i(x_{i_j}, y, z) \cap -intC_i(z) = \emptyset,$$

(iii) Ψ_i is said to be $C_i(z) - FC-$ diagonal quasi-convex of strong type 1 in first argument if each $N_i = \{x_{i,0}, \dots, x_{i,n}\} \in \langle X_i\rangle$ and for each $z \in X$ with $z_i \in FC(N_i)$, there exists $j \in \{0, \dots, k\}$ such that

$$\Psi_i(x_{i_j}, y, z) \subset C_i(z).$$

(iii) Ψ_i is said to be $C_i(z) - FC-$ diagonal quasi-convex of strong type 2 in first argument if each $N_i = \{x_{i,0}, \dots, x_{i,n}\} \in \langle X_i\rangle$ and for each $z \in X$ with $z_i \in FC(N_i)$, there exists $j \in \{0, \dots, k\}$ such that

$$\Psi_i(x_{i_j}, y, z) \cap C_i(z) \neq \emptyset.$$

Now, we establish an existence result for a solution of systems of generalized vector quasi-equilibrium problems (SGVQEP) as follows :

Theorem 3.2. *Let K and H be nonempty compact subsets of X and Y, respectively, and for each $i \in I, C_i(z)$ be closed convex cone with nonempty interior. Suppose that for each $i \in I$, the following condition are satisfied;*

(i) *for each $(y, z) \in Y \times X, T_i(y, z)$ and $A_i(y, z)$ are nonempty $FC-$subspaces of Y_i and X_i, respectively;*
(ii) *for each $(u_i, v_i) \in Y_i \times X_i, T_i^{-1}(u_i)$ and $A_i^{-1}(v_i)$ are compactly open in $Y \times X$;*
(iii) *the mapping $z \mapsto intC_i(z)$ has an open graph and for each $x_i \in X_i$, the mapping $(y, z) \mapsto \Psi_i(x_i, y, z)$ is upper semicontinuous on each compact subsets of $Y \times X$ with nonempty compactly closed values;*
(iv) *for each $z \in X, \Psi_i : X_i \times Y \times X \to 2^{Z_i}$ is $C_i(z)-FC-$diagonal quasi-convex of weak type 1 in first argument;*
(v) *the set $W_i = \{(y, z) \in Y \times X : y_i \in T_i(y, z), z_i \in A_i(y, z)\}$ is compactly closed in $Y \times X$;*
(vi) *there exists nonempty compact subset $H \times K$ of $Y \times X$, and for each $M_i \times N_i \in \langle Y_i \times X_i\rangle$, there exists compact FC-subspace $L_{M_i} \times L_{N_i}$ containing $M_i \times N_i$ such that for each $(y, z) \in Y \times X \setminus H \times K$, there exist $i \in I, \bar{u}_i \in T_i(y, z) \cap L_{M_i}$ and $\bar{v}_i \in A_i(y, z) \cap L_{N_i}$ satisfying $\Psi_i(\bar{v}_i, y, z) \subset -intC_i(z)$.*

Then the solution set M_1 of SGVQEP(1) is nonempty and compact in $H \times K$, where $M_1 = \{(y,z) \in H \times K : y_i \in T_i(y,z), z_i \in A_i(y,z)$ *and* $\Psi_i(x_i,y,z) \nsubseteq -intC_i(z), \forall x_i \in A_i(y,z), i \in I\}$.

Proof. **Step I.** Show that M_1 is nonempty.
For each $i \in I$, we define a set-valued mapping $P_i : Y \times X \to 2^{X_i}$ by

$$P_i(y,z) = \{x_i \in X_i : \Psi_i(x_i,y,z) \subset -intC_i(z)\}, \forall (y,z) \in Y \times X.$$

We now, show that for each $i \in I$ and $(y,z) \in Y \times X$,

$$z_i = \pi_i(z) \notin FC(P_i(y,z)). \tag{3.1}$$

If it is false, then there exist $i \in I$ and $(\bar{y},\bar{z}) \in Y \times X$ such that $\bar{z}_i = \pi_i(\bar{z}) \in FC(P_i(\bar{y},\bar{z}))$. Hence by Lemma 2.3, there exists $N_i = \{x_{i,o},\dots,x_{i,n}\} \in \langle P_i(\bar{y},\bar{z})\rangle$ such that $\bar{z}_i = \pi_i(\bar{z}) \in FC(N_i)$. Thus, we have

$$\Psi_i(x_{i_j},\bar{y},\bar{z}) \subset -intC_i(\bar{z}), \forall j = 0,\dots,n.$$

By (iv) and Definition 3.1.(i), there exists $j \in \{0,\dots,n\}$ such that

$$\Psi_i(x_{i_j},\bar{y},\bar{z}) \nsubseteq -intC_i(\bar{z}), \forall j = 0,\dots,n,$$

which is a contradiction. Then, for each $i \in I$ and $(y,z) \in Y \times X, z_i = \pi_i(z) \notin FC(P_i(y,z))$. Hence by condition (iii) and Lemma 2.9, for each $i \in I$ and $x_i \in X_i$, the set

$$P_i^{-1}(x_i) = \{(y,z) \in Y \times X : \Psi_i(x_i,y,z) \subset -intC_i(z)\}$$

is compactly open in $\in Y \times X$. From Lemma 2.4 that $(FC(P_i))^{-1}(x_i)$ is also compactly open in $\in Y \times X$ for each $x_i \in X_i$. By Lemma 2.5, it follows $i \in I, Y_i \times X_i$ is a FC-space and $Y \times X$ is also a FC-space for all $i \in I$.

Next, for each $i \in I$, we define a set-valued mapping $G_i : Y \times X \to 2^{Y_i \times X_i}$ by

$$G_i(y,z) = \begin{cases} T_i(y,z) \times [A_i(y,z) \cap FC(P_i(y,z))], \ if \ (y,z) \in W_i, \\ T_i(y,z) \times A_i(y,z), \ if (y,z) \notin W_i. \end{cases}$$

By condition (i), for each $i \in I$ and $(y,z) \in Y \times X$, $G_i(y,z)$ is an FC-subspace of $Y_i \times X_i$. From the definition of W_i and (3.2), $(y_i,z_i) \notin G_i(y,z)$, for each $i \in I$ and $(y,z) \in Y \times X$.

Then, for each $i \in I$ and $(u_i,v_i) \in Y_i \times X_i$, we have

$$\begin{aligned} G_i^{-1}(u_i,v_i) &= [T_i^{-1}(u_i) \bigcap A_i^{-1}(v_i) \bigcap (FC(P_i))^{-1}(v_i)] \\ &\quad \bigcup [(Y \times X \backslash W_i) \bigcap T_i^{-1}(u_i) \bigcap A_i^{-1}(v_i)]. \end{aligned}$$

Since $(FC(P_i))^{-1}(v_i)$ is compactly open in $Y \times X$ for each $v_i \in X_i$, it follows by the condition (ii) that $G_i^{-1}(u_i,v_i)$ is also compactly open in $Y \times X$. By (vi), for each $H_i = M_i \times N_i \in \langle Y_i \times X_i\rangle$ there exists compact FC-subspace $L_{H_i} = L_{M_i} \times L_{N_i}$ of $Y_i \times X_i$ containing H_i such that

$$G_i(y,z) \cap L_{H_i} \neq \emptyset.$$

Thus all conditions of Lemma 2.6 are satisfied. Hence by Lemma 2.6, there exists $(\hat{y},\hat{z}) \in H \times K$ such that $G_i(\hat{y},\hat{z}) = \emptyset$ for each $i \in I$. If $(\hat{y},\hat{z}) \notin W_j$ for some $j \in I$, then either $T_j(\hat{y},\hat{z}) = \emptyset$ or $A_j(\hat{y},\hat{z}) = \emptyset$, which contradicts the condition (i). Therefore, $(\hat{y},\hat{z}) \in W_i$ for all $i \in I$. This implies that for each $i \in I, \hat{y}_i \in T_i(\hat{y},\hat{z}), \hat{z}_i \in A_i(\hat{y},\hat{z})$ and $A_i(\hat{y},\hat{z}) \cap FC(P_i(\hat{y},\hat{z})) = \emptyset$ and hence $A_i(\hat{y},\hat{z}) \cap P_i(\hat{y},\hat{z}) = \emptyset$. Therefore, for each $i \in I, \hat{y}_i \in T_i(\hat{y},\hat{z}), \hat{z}_i \in A_i(\hat{y},\hat{z})$ and

$$\Psi_i(x_i,\hat{y},\hat{z}) \nsubseteq -intC_i(\hat{z}), \forall x_i \in A_i(\hat{y},\hat{z}).$$

Hence, $(\hat{y}, \hat{z}) \in M_1$,and M_1 is nonempty.
Step II. Show that M_1 is compact.
By condition (iii) and Lemma 2.9, we note that, for each $i \in I$ and $v_i \in X_i$, the set

$$\{(y, z) \in Y \times X : \Psi_i(v_i, y, z) \nsubseteq -intC_i(z)\}$$

is compactly closed in $Y \times X$. This implies that the set

$$\{(y, z) \in H \times K : \Psi_i(x_i, y, z) \nsubseteq -intC_i(z), \forall x_i \in A_i(y, z)\}$$

is closed in $H \times K$, for all $i \in I$. By condition (v), the set

$$W_i = \{(y, z) \in H \times K : y_i \in T_i(y, z), z_i \in A_i(y, z)\}$$

is also closed in $H \times K$ for each $i \in I$. It follows that M_1 is closed in $H \times K$. Hence $H \times K$ is compact in $Y \times X$ and therefore M_1 is nonempty and compact . This completes the proof. □

Theorem 3.3. *Let K and H be nonempty compact subsets of X and Y, respectively, and for each $i \in I, C_i(z)$ be closed convex cone with nonempty interior. Suppose that for each $i \in I$, the following condition are satisfied;*

- (i) *for each $(y, z) \in Y \times X, T_i(y, z)$ and $A_i(y, z)$ are nonempty $FC-$subspaces of Y_i and X_i, respectively;*
- (ii) *for each $(u_i, v_i) \in Y_i \times X_i, T_i^{-1}(u_i)$ and $A_i^{-1}(v_i)$ are compactly open in $Y \times X$;*
- (iii) *the mapping $z \mapsto intC_i(z)$ has an open graph and for each $x_i \in X_i$, the mapping $(y, z) \mapsto \Psi_i(x_i, y, z)$ is lower semicontinuous on each compact subsets of $Y \times X$ with nonempty compactly closed values;*
- (iv) *for each $z \in X, \Psi_i : X_i \times Y \times X \to 2^{Z_i}$ is $C_i(z)-FC-$diagonal quasi-convex of weak type 2 in first argument;*
- (v) *the set $W_i = \{(y, z) \in Y \times X : y_i \in T_i(y, z), z_i \in A_i(y, z)\}$ is compactly closed in $Y \times X$;*
- (vi) *there exists nonempty compact subset $H \times K$ of $Y \times X$, and for each $M_i \times N_i \in \langle Y_i \times X_i \rangle$, there exists compact FC-subspace $L_{M_i} \times L_{N_i}$ containing $M_i \times N_i$ such that for each $(y, z) \in Y \times X \setminus H \times K$, there exist $i \in I, \bar{u}_i \in T_i(y, z) \cap L_{M_i}$ and $\bar{v}_i \in A_i(y, z) \cap L_{N_i}$ satisfying $\Psi_i(\bar{v}_i, y, z) \cap (-intC_i(z)) \neq \emptyset$.*

Then the solution set M_2 of SGVQEP(2) is nonempty and compact in $H \times K$, where $M_2 = \{(y, z) \in H \times K : y_i \in T_i(y, z), z_i \in A_i(y, z)$ and $\Psi_i(x_i, y, z) \cap (-intC_i(z)) = \emptyset, \forall x_i \in A_i(y, z), i \in I\}$.

Proof. For each $i \in I$, we define a set-valued mapping $P_i : Y \times X \to 2^{X_i}$ by

$$P_i(y, z) = \{x_i \in X_i : \Psi_i(x_i, y, z) \cap (-intC_i(z)) \neq \emptyset\}, \forall (y, z) \in Y \times X.$$

We now show that, for each $i \in I$ and $(y, z) \in Y \times X$,

$$z_i = \pi_i(z) \notin FC(P_i(y, z)). \tag{3.2}$$

If it is false, then there exist $i \in I$ and $(\bar{y}, \bar{z}) \in Y \times X$ such that $\bar{z}_i = \pi_i(\bar{z}) \in FC(P_i(\bar{y}, \bar{z}))$. Hence by Lemma 2.3, there exists $N_i = \{x_{i,o}, \ldots, x_{i,n}\} \in \langle P_i(\bar{y}, \bar{z})\rangle$, such that $\bar{z}_i = \pi_i(\bar{z}) \in FC(N_i)$. Thus, we note that

$$\Psi_i(x_{i_j}, \bar{y}, \bar{z}) \subset -intC_i(\bar{z}), \forall j = 0, \ldots, n.$$

It follows by (iv) and Definition 3.1.(ii) that there exists $j \in \{0, \ldots, n\}$ such that

$$\Psi_i(x_{i_j}, \bar{y}, \bar{z}) \nsubseteq -intC_i(\bar{z}), \forall j = 0, \ldots, n,$$

which is a contradiction. Hence, for each $i \in I$ and $(y,z) \in Y \times X, z_i = \pi_i(z) \notin FC(P_i(y,z))$. By condition (iii) and Lemma 2.8, we note that, the set

$$P_i^{-1}(x_i) = \{(y,z) \in Y \times X : \Psi_i(x_i,y,z) \cap (-intC_i(z)) \neq \emptyset\}$$

is compactly open in $\in Y \times X$ for all $i \in I$ and $x_i \in X_i$. It follows from Lemma 2.4 that $(FC(P_i))^{-1}(x_i)$ is also compactly open in $\in Y \times X$ for each $x_i \in X_i$. Hence, by Lemma 2.5, for each $i \in I, Y_i \times X_i$ is a FC-space and $Y \times X$ is also an FC-space. By the similar argument as in the proof of Theorem 3.2, we obtain the desired result. □

Theorem 3.4. *Let K and H be nonempty compact subsets of X and Y, respectively, and for each $i \in I, C_i(z)$ be closed convex cone with nonempty interior. Suppose that for each $i \in I$,*

(i) *for each $(y,z) \in Y \times X, T_i(y,z)$ and $A_i(y,z)$ are nonempty $FC-$subspaces of Y_i and X_i, respectively;*
(ii) *for each $(u_i,v_i) \in Y_i \times X_i, T_i^{-1}(u_i)$ and $A_i^{-1}(v_i)$ are compactly open in $Y \times X$;*
(iii) *the mapping $z \mapsto intC_i(z)$ has an open graph and for each $x_i \in X_i$, the mapping $(y,z) \mapsto \Psi_i(x_i,y,z)$ is lower semicontinuous on each compact subsets of $Y \times X$ with nonempty compactly closed values;*
(iv) *for each $z \in X, \Psi_i : X_i \times Y \times X \to 2^{Z_i}$ is $C_i(z)-FC-$diagonal quasi-convex of strong type 1 in first argument;*
(v) *the set $W_i = \{(y,z) \in Y \times X : y_i \in T_i(y,z), z_i \in A_i(y,z)\}$ is compactly closed in $Y \times X$;*
(vi) *there exists nonempty compact subset $H \times K$ of $Y \times X$, and for each $M_i \times N_i \in \langle Y_i \times X_i \rangle$, there exists compact FC-subspace $L_{M_i} \times L_{N_i}$ containing $M_i \times N_i$ such that for each $(y,z) \in Y \times X \setminus H \times K$, there exist $i \in I, \bar{u}_i \in T_i(y,z) \cap L_{M_i}$ and $\bar{v}_i \in A_i(y,z) \cap L_{N_i}$ satisfying $\Psi_i(\bar{v}_i,y,z) \nsubseteq -C_i(z)$.*

Then the solution set M_3 of SGVQEP(3) is nonempty and compact in $H \times K$, where $M_3 = \{(y,z) \in H \times K : y_i \in T_i(y,z), z_i \in A_i(y,z)$ and $\Psi_i(x_i,y,z) \subset -C_i(z), \forall x_i \in A_i(y,z), i \in I\}$.

Proof. For each $i \in I$, we define a set-valued mapping $P_i : Y \times X \to 2^{X_i}$ by

$$P_i(y,z) = \{x_i \in X_i : \Psi_i(x_i,y,z) \nsubseteq -C_i(z)\}, \forall (y,z) \in Y \times X.$$

We now show that, for each $i \in I$ and $(y,z) \in Y \times X$,

$$z_i = \pi_i(z) \notin FC(P_i(y,z)). \tag{3.3}$$

If it is false, then there exist $i \in I$ and $(\bar{y},\bar{z}) \in Y \times X$ such that $\bar{z}_i = \pi_i(\bar{z}) \in FC(P_i(\bar{y},\bar{z}))$. Hence by Lemma 2.3, there exists $N_i = \{x_{i,o},\dots,x_{i,n}\} \in \langle P_i(\bar{y},\bar{z}) \rangle$, such that $\bar{z}_i = \pi_i(\bar{z}) \in FC(N_i)$. Thus, we note that

$$\Psi_i(x_{i_j},\bar{y},\bar{z}) \nsubseteq -C_i(\bar{z}), \forall j = 0,\dots,n.$$

It follows (iv) and Definition 3.1.(iii) that there exists $j \in \{0,\dots,n\}$ such that

$$\Psi_i(x_{i_j},\bar{y},\bar{z}) \subset -C_i(\bar{z}), \forall j = 0,\dots,n,$$

which is a contradiction. Hence, for each $i \in I$ and $(y,z) \in Y \times X, z_i = \pi_i(z) \notin FC(P_i(y,z))$. Then by condition (iii) and Lemma 2.8, for each $i \in I$ and $x_i \in X_i$, the set

$$P_i^{-1}(x_i) = \{(y,z) \in Y \times X : \Psi_i(x_i,y,z) \nsubseteq -C_i(z)\}$$

is compactly open in $\in Y \times X$. It follows from Lemma 2.4 that $(FC(P_i))^{-1}(x_i)$ is also compactly open in $\in Y \times X$ for each $x_i \in X_i$. Hence, by Lemma 2.5, for each $i \in I, Y_i \times X_i$ is a FC-space and $Y \times X$ is also an FC-space. By the similar argument as in the proof of Theorem 3.2, we obtain the desired result. □

Definition 3.5. For each $i \in I$ and $(y,z) \in Y \times X, \Psi_i : X_i \times Y \times X \to 2^{Z_i}$ is said to be

(i) $C_i(z) - FC-$ quasi-convex in first argument if each $(y,z) \in Y \times X$, $N_i = \{x_{i,0}, \ldots, x_{i,n}\} \in \langle X_i \rangle, \{x_{i,i_0}, \ldots, x_{i,i_k}\} \subset N_i$ and $x_i^* \in \varphi'_{N_i}(\Delta_k)$, there exists $j \in \{0, \ldots, k\}$ such that

$$\Psi_i(x_{i,i_j}, y, z) \subset \Psi_i(x_i^*, y, z) + C_i(z),$$

(ii) $C_i(z) - FC-$ quasi-convex-like in first argument if each $(y,z) \in Y \times X$, $N_i = \{x_{i,0}, \ldots, x_{i,n}\} \in \langle X_i \rangle, \{x_{i,i_0}, \ldots, x_{i,i_k}\} \subset N_i$ and $x_i^* \in \varphi'_{N_i}(\Delta_k)$, there exists $j \in \{0, \ldots, k\}$ such that

$$\Psi_i(x_i^*, y, z) \subset \Psi_i(x_{i,i_j}, y, z) - C_i(z).$$

Form Ding [23], we give the following lemma;

Lemma 3.6. [23] *For each $i \in I$, let Z_i be a topological vector space and $C_i : X \to 2^{Z_i}$ be a set-valued mapping, such that for each $z \in X, C_i(z)$ is closed convex cone in Z_i with nonempty interior. If for each $i \in I, (y,z) \in Y \times X, \Psi_i : X_i \times Y \times X \to 2^{Z_i}$ is $C_i(z) - FC-$quasiconvex-like in first argument, then, the set*

$$\{x_i \in X_i : \Psi_i(x_i, y, z) \nsubseteq -intC_i(z)\}$$

is FC-subspace of X_i.

Lemma 3.7. [23] *For each $i \in I$, let Z_i be a topological vector space and $C_i : X \to 2^{Z_i}$ be a set-valued mapping, such that for each $z \in X, C_i(z)$ is closed convex cone in Z_i with nonempty interior. If for each $i \in I, (y,z) \in Y \times X, \Psi_i : X_i \times Y \times X \to 2^{Z_i}$ is $C_i(z) - FC-$quasiconvex in first argument, then, the set*

$$\{x_i \in X_i : \Psi_i(x_i, y, z) \cap -intC_i(z) = \emptyset\}$$

is FC-subspace of X_i.

Lemma 3.8. [23] *For each $i \in I$, let Z_i be a topological vector space and $C_i : X \to 2^{Z_i}$ be a set-valued mapping, such that for each $z \in X, C_i(z)$ is closed convex cone in Z_i with nonempty interior. If for each $i \in I, (y,z) \in Y \times X, \Psi_i : X_i \times Y \times X \to 2^{Z_i}$ is $C_i(z) - FC-$quasiconvex-like in first argument, then, the set*

$$\{x_i \in X_i : \Psi_i(x_i, y, z) \subset -C_i(z)\}$$

is FC-subspace of X_i.

Theorem 3.9. *Let K and H be nonempty compact subsets of X and Y, respectively, and for each $i \in I, C_i(z)$ be closed convex cone with nonempty interior. Suppose that for each $i \in I$, the following condition are satisfied;*

(i) *for each $(y,z) \in Y \times X, T_i(y,z)$ and $A_i(y,z)$ are nonempty $FC-$subspaces of Y_i and X_i, respectively;*
(ii) *for each $(u_i, v_i) \in Y_i \times X_i, T_i^{-1}(u_i)$ and $A_i^{-1}(v_i)$ are compactly open in $Y \times X$;*
(iii) *the mapping $z \mapsto intC_i(z)$ has an open graph and for each $x_i \in X_i$, the mapping $(y,z) \mapsto \Psi_i(x_i, y, z)$ is upper semicontinuous on each compact subsets of $Y \times X$ with nonempty compactly closed values;*
(iv) *for each $z \in X, \Psi_i : X_i \times Y \times X \to 2^{Z_i}$ is $C_i(z) - FC-$ quasi-convex-like in first argument and for each $(y,z) \in Y \times X, \Psi_i(x_i, y, z) \nsubseteq -intC_i(z)$;*

(v) *the set* $W_i = \{(y, z) \in Y \times X : y_i \in T_i(y, z), z_i \in A_i(y, z)\}$ *is compactly closed in* $Y \times X$*;*
(vi) *there exists nonempty compact subset* $H \times K$ *of* $Y \times X$*, and for each* $M_i \times N_i \in \langle Y_i \times X_i \rangle$*, there exists compact FC-subspace* $L_{M_i} \times L_{N_i}$ *containing* $M_i \times N_i$ *such that for each* $(y, z) \in Y \times X \setminus H \times K$*, there exist* $i \in I, \bar{u_i} \in T_i(y, z) \cap L_{M_i}$ *and* $\bar{v_i} \in A_i(y, z) \cap L_{N_i}$ *satisfying* $\Psi_i(\bar{v_i}, y, z) \subset -intC_i(z)$*.*

Then the solution set M_1 *of SGVQEP(1) is nonempty and compact in* $H \times K$*, where* $M_1 = \{(y, z) \in H \times K : y_i \in T_i(y, z), z_i \in A_i(y, z)$ *and* $\Psi_i(x_i, y, z) \nsubseteq -intC_i(z), \forall x_i \in A_i(y, z), i \in I\}$*.*

Proof. We first show that, Ψ_i is $C_i(z) - FC-$diagonal quasi-convex of SK-type 1 in first argument for all $i \in I$ and $z \in X$. If it is false, then there exist $i \in I$, $N_i = \{x_{i,0}, \ldots, x_{i,n}\} \in \langle X_i \rangle$ and for each $\bar{z} \in X$ with $\bar{z_i} \in FC(N_i)$, such that $\Psi_i(x_{i_j}, y, z) \subset -intC_i(z)$, for all $j \in \{0, \ldots, k\}$. Hence, we obtain $N_i \subset P_i(y, \bar{z})$. It follows from (iv) and Lemma 3.6 that for each $i \in I$ and $(y, \bar{z}) \in Y \times X$, $P_i(y, \bar{z})$ is an FC-subspace of X_i. Then we have $\bar{z_i} \in FC(N_i) \subset P_i(y, \bar{z})$, which contradicts the fact that for each $(y, z) \in Y \times X, z_i \notin P_i(y, z)$. Therefore, for each $i \in I$ and $z \in X$, Ψ_i is $C_i(z) - FC-$diagonal quasi-convex of SK-type 1 in first argument. Thus all conditions of Theorem 3.2 for the SGVQEP(1) are satisfied. Hence the conclusion of Theorem 3.9 hold from Theorem 3.2. This completes the proof. □

Theorem 3.10. *Let* K *and* H *be nonempty compact subsets of* X *and* Y*, respectively, and for each* $i \in I, C_i(z)$ *be closed convex cone with nonempty interior. Suppose that for each* $i \in I$*, the following condition are satisfied;*

(i) *for each* $(y, z) \in Y \times X, T_i(y, z)$ *and* $A_i(y, z)$ *are nonempty* $FC-$*subspaces of* Y_i *and* X_i*, respectively;*
(ii) *for each* $(u_i, v_i) \in Y_i \times X_i, T_i^{-1}(u_i)$ *and* $A_i^{-1}(v_i)$ *are compactly open in* $Y \times X$*;*
(iii) *the mapping* $z \mapsto intC_i(z)$ *has an open graph and for each* $x_i \in X_i$*, the mapping* $(y, z) \mapsto \Psi_i(x_i, y, z)$ *is lower semicontinuous on each compact subsets of* $Y \times X$ *with nonempty compactly closed values;*
(iv) *for each* $z \in X, \Psi_i : X_i \times Y \times X \to 2^{Z_i}$ *is* $C_i(z) - FC-$ *quasi-convex in first argument and for each* $(y, z) \in Y \times X, \Psi_i(x_i, y, z) \cap (-intC_i(z)) = \emptyset$*;*
(v) *the set* $W_i = \{(y, z) \in Y \times X : y_i \in T_i(y, z), z_i \in A_i(y, z)\}$ *is compactly closed in* $Y \times X$*;*
(vi) *there exists nonempty compact subset* $H \times K$ *of* $Y \times X$*, and for each* $M_i \times N_i \in \langle Y_i \times X_i \rangle$*, there exists compact FC-subspace* $L_{M_i} \times L_{N_i}$ *containing* $M_i \times N_i$ *such that for each* $(y, z) \in Y \times X \setminus H \times K$*, there exist* $i \in I, \bar{u_i} \in T_i(y, z) \cap L_{M_i}$ *and* $\bar{v_i} \in A_i(y, z) \cap L_{N_i}$ *satisfying* $\Psi_i(\bar{v_i}, y, z) \cap (-intC_i(z)) \neq \emptyset$*.*

Then the solution set M_2 *of SGVQEP(2) is nonempty and compact in* $H \times K$*, where* $M_2 = \{(y, z) \in H \times K : y_i \in T_i(y, z), z_i \in A_i(y, z)$ *and* $\Psi_i(x_i, y, z) \cap (-intC_i(z)) = \emptyset, \forall x_i \in A_i(y, z), i \in I\}$*.*

Proof. We first show that ,Ψ_i is $C_i(z) - FC-$diagonal quasi-convex of SK-type 2 in first argument, for each $i \in I$ and $z \in X$. If it is false, then, there exist $i \in I$, $N_i = \{x_{i,0}, \ldots, x_{i,n}\} \in \langle X_i \rangle$ and for each $\bar{z} \in X$ with $\bar{z_i} \in FC(N_i)$, such that $\Psi_i(x_{i_j}, y, z) \cap -intC_i(z) \neq \emptyset$ for all $j \in \{0, \ldots, k\}$. Hence, we obtain $N_i \subset P_i(y, \bar{z})$. It follows from (iv) and Lemma 3.7 that for each $i \in I$ and $(y, \bar{z}) \in Y \times X$, $P_i(y, \bar{z})$ is an FC-subspace of X_i. Then we have $\bar{z_i} \in FC(N_i) \subset P_i(y, \bar{z})$, which contradicts the fact that for each $(y, z) \in Y \times X, z_i \notin P_i(y, z)$. Therefore, for each $i \in I$ and

$z \in X$, Ψ_i is $C_i(z) - FC-$diagonal quasi-convex of SK-type 2 in first argument. Thus all conditions of Theorem 3.3 for the SGVQEP(2) are satisfied. Hence the conclusion of Theorem 3.10 hold from Theorem 3.3. This completes the proof. □

By applying Lemma 2.6, Lemma 3.8 and the similar argument as in the proof of Theorem 3.9-3.10, we can easily prove the following results.

Theorem 3.11. *Let K and H be nonempty compact subsets of X and Y, respectively, and for each $i \in I, C_i(z)$ be closed convex cone with nonempty interior. Suppose that for each $i \in I$,*

- (i) *for each $(y, z) \in Y \times X, T_i(y, z)$ and $A_i(y, z)$ are nonempty $FC-$subspaces of Y_i and X_i, respectively;*
- (ii) *for each $(u_i, v_i) \in Y_i \times X_i, T_i^{-1}(u_i)$ and $A_i^{-1}(v_i)$ are compactly open in $Y \times X$;*
- (iii) *the mapping $z \mapsto intC_i(z)$ has an open graph and for each $x_i \in X_i$, the mapping $(y, z) \mapsto \Psi_i(x_i, y, z)$ is lower semicontinuous on each compact subsets of $Y \times X$ with nonempty compactly closed values;*
- (iv) *for each $z \in X, \Psi_i : X_i \times Y \times X \to 2^{Z_i}$ is $C_i(z) - FC-$quasi-convex-like in first argument and for each $(y, z) \in Y \times X, \Psi_i(x_i, y, z) \subset -C_i(z)$;*
- (v) *the set $W_i = \{(y, z) \in Y \times X : y_i \in T_i(y, z), z_i \in A_i(y, z)\}$ is compactly closed in $Y \times X$;*
- (vi) *there exists nonempty compact subset $H \times K$ of $Y \times X$, and for each $M_i \times N_i \in \langle Y_i \times X_i \rangle$, there exists compact FC-subspace $L_{M_i} \times L_{N_i}$ containing $M_i \times N_i$ such that for each $(y, z) \in Y \times X \setminus H \times K$, there exist $i \in I, \bar{u}_i \in T_i(y, z) \cap L_{M_i}$ and $\bar{v}_i \in A_i(y, z) \cap L_{N_i}$ satisfying $\Psi_i(\bar{v}_i, y, z) \not\subseteq -C_i(z)$.*

Then the solution set M_3 of SGVQEP(3) is nonempty and compact in $H \times K$, where $M_3 = \{(y, z) \in H \times K : y_i \in T_i(y, z), z_i \in A_i(y, z)$ and $\Psi_i(x_i, y, z) \subset -C_i(z), \forall x_i \in A_i(y, z), i \in I\}$.

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