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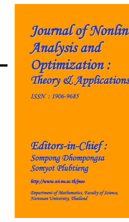

T-BEST APPROXIMATION IN FUZZY AND INTUITIONISTIC FUZZY METRIC SPACES

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ABSTRACT. In this paper we introduce and extend some notions in [18] and introduce a notion of t-best approximatively compact sets, t-best approximation points, t-proximinal sets and t-boundedly compact sets in both fuzzy and intuitionistic fuzzy metric spaces. The results obtained in this paper are related to the corresponding results in metric spaces and fuzzy metric spaces and fuzzy normed space. Many examples are also given.

KEYWORDS : Best approximation; Topology; Intuitionistic fuzzy metric spaces.

1. INTRODUCTION AND PRELIMINARIES

It is well known that the notion of fuzzy metric spaces plays a fundamental role in fuzzy topology, so many authors have introduced and studied several notions of fuzzy metric spaces from different points of view. In particular, following Menger [9], Kramosil and Michalek [8] generalized the concept of probabilistic metric space and studied an interesting notion of fuzzy metric space with the help of continuous t-norm. Later on, in order to construct a Hausdorff topology on the fuzzy metric space George and Veeramani [6] modified the concept of fuzzy metric space introduced by Kramosil and Michalek and obtained several classical theorems on this new structure. Actually, this topology is first countable and metrizable [3]. Further results in the topology of fuzzy metric spaces, in the sense of [6] may be found in [2, 3, 5, 7, 11, 15]. Park [10] extended the notion of fuzzy metric space proposed by George and Veeramani [6] and introduced the notion of intuitionistic fuzzy metric space which is based both on the idea of intuitionistic fuzzy set and the concept

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of fuzzy metric spaces. The topology generated by intuitionistic fuzzy metric space coincides with the topology generated by fuzzy metric space and, hence, topological results for intuitionistic fuzzy metric space are immediate consequences of the corresponding for fuzzy metric space. Some results in the topology of the intuitionistic fuzzy metric spaces may be found in [4, 10, 12].

Best approximation in fuzzy metric spaces has been discussed by Veeramani in [18]. In this paper , we start with the definitions of intuitionistic fuzzy metric spaces [10] and fuzzy normed spaces [13] and conclude some useful results, to be used in the next section. In section 2, we define the notion of t-approximatively compact sets in fuzzy and intuitionistic fuzzy metric spaces and introduce the notions of t-proximinal sets, t-boundedly compact sets, and t-best approximation points, this notions are the generalization of [13, 18]. For this notions we also point out some results about the relationship between metric spaces, fuzzy metric spaces and intuitionistic fuzzy metric spaces. To define the intuitionistic fuzzy metric space we have to state several concepts as follows:

Definition 1.1. [14]. A binary operation $* : [0,1] \times [0,1] \longrightarrow [0,1]$ is continuous t-norm if $*$ satisfies the following conditions:

- **(a):** $*$ is commutative and associative;
- **(b):** $*$ is continuous;
- **(c):** $a * 1 = a$ for all $a \in [0,1]$;
- **(d):** $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ and $a, b, c, d \in [0,1]$.

Definition 1.2. [14]. A binary operation $\Diamond : [0,1] \times [0,1] \longrightarrow [0,1]$ is continuous t-conorm if $\Diamond$ satisfies the following conditions:

- **(a):** $\Diamond$ is commutative and associative;
- **(b):** $\Diamond$ is continuous;
- **(c):** $a \Diamond 0 = a$ for all $a \in [0,1]$;
- **(d):** $a \Diamond b \leq c \Diamond d$ whenever $a \leq c$ and $b \leq d$ and $a, b, c, d \in [0,1]$.

Definition 1.3. [10]. A 5-tuple $(X, M, N, *, \Diamond)$ is said to be an *intuitionistic fuzzy metric space* if X is a arbitrary set, $*$ is a continuous t-norm, $\Diamond$ is a continuous t-conorm and M, N are fuzzy sets on $X \times X \times (0, \infty)$, satisfying the following conditions, for all $x, y, z \in X, s, t > 0$

- **(a):** $M(x, y, t) + N(x, y, t) \leq 1$;
- **(b):** $M(x, y, t) > 0$;
- **(c):** $M(x, y, t) = 1$ if and only if $x = y$;
- **(d):** $M(x, y, t) = M(y, x, t)$;
- **(e):** $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
- **(f):** $M(x, y, .) : (0, \infty) \to (0, 1]$ is continuous;
- **(g):** $N(x, y, t) > 0$;
- **(h):** $N(x, y, t) = 0$ if and only if $x = y$;
- **(i):** $N(x, y, t) = N(y, x, t)$;
- **(j):** $N(x, y, t) \Diamond N(y, z, s) \geq N(x, z, t + s)$;
- **(k):** $N(x, y, .) : (0, \infty) \to (0, 1]$ is continuous.

The functions $M(x, y, t)$, $N(x, y, t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively. A *fuzzy metric space* is a triple $(X, M, *)$ such that conditions (b)-(f) are satisfied[6].

Definition 1.4. [13] The 3-tuple $(X, N, *)$ is said to be a fuzzy normed space if X is a vector space, $*$ is a continuous t-norm and N is a fuzzy set on $X \times (0, 1)$ satisfying the following conditions for every $x, y \in X$ and $t, s > 0$,

(a): $N(x,t) > 0$;
(b): $N(x,t) = 1$ if and only if $x = 0$;
(c): $N(\alpha x, t) = N(x, t/|\alpha|)$, for all $\alpha \neq 0$;
(d): $N(x,t) * N(y,s) \leq N(x+y, t+s)$;
(e): $N(x,.) : (0,\infty) \to [0,1]$ is continuous;
(f): $\lim_{t\to\infty} N(x,t) = 1$

Remark 1.5. [13] Every fuzzy normed space $(X,N,*)$ induces a fuzzy metric space $(X,M,*)$ by defining $M(x,y,t) = N(x-y,t)$ and is therefore a topological space

If $(X,M,*)$ is a fuzzy metric space then George and Veeramani proved [6] that every fuzzy metric spaces $(X,M,*)$ generates a Hausdorff first countable topology τ_M on X which has as a base the family of open balls of the form $\{B_M(x,r,t); x \in X, r \in (0,1), t > 0\}$, where $B_M(x,r,t) = \{y \in X : M(x,y,t) > 1-r\}$. Furthermore, for each x in X, $\{B(x,1/n,1/n); n \in \mathbb{N}\}$ is a neighborhood local base at x for the topology τ_M, so τ_M is the first countable. Recently, George and Romaguera proved [3] that if $(X,M,*)$ is a fuzzy metric space, then $\{U_n; n \in \mathbb{N}\}$ is a base for a uniformity $\mathcal{U}_n$ on X compatible with τ_M, where $U_n = \{(x,y) \in X \times X; M(x,y,1/n) > 1-(1/n)\}$ for all $n \in \mathbb{N}$. Therefore, $(X,M,*)$ is a metrizable space. Analogously, Saadati and Park [12] proved for intuitionistic fuzzy metric spaces $(X,M,N,*,\diamondsuit)$ that the topology $\tau_{(M,N)}$ on X for which open balls are of the form $\{B_{(M,N)}(x,r,t); x \in X, r \in (0,1), t > 0\}$ where $B_{(M,N)}(x,r,t) = \{y \in X : M(x,y,t) > 1-r, N(y,x,t) < r\}$ is metrizable.

Remark 1.6. [4, Proposition 1] For each $x \in X$, $r \in (0,1)$ and $t > 0$, we have $B_M(x,r,t) = B_{(M,N)}(x,r,t)$, thus, the two topologies τ_M and $\tau_{(M,N)}$ are equivalent in X, and the results obtained in the topology of intuitionistic fuzzy metric spaces become immediate consequences of the corresponding results of the topology of fuzzy metric spaces.

Remark 1.7. In a fuzzy metric space $(X,M,*)$ for each x in X, $0 < r < 1$ and $t > 0$, the set $B_M[x,r,t]$ defined as

$$B_M[x,r,t] = \{y \in X : M(x,y,t) \geq 1-r\}$$

is a closed set [6]. By a similar proof [18, Preposition.1] we deduce for each $x \in X$, $0 < r < 1$ and $t > 0$ in an intuitionistic fuzzy metric space $(X,M,N,*,\diamondsuit)$, the set $B_{(M,N)}[x,r,t]$ defined as

$$B_{(M,N)}[x,r,t] = \{y \in X : M(x,y,t) \geq 1-r, N(y,x,t) \leq r\}$$

is equal to $B_M[x,r,t]$. Consequently, the set $B_{(M,N)}[x,r,t]$ is a closed set in the topology $\tau_{(M,N)}$ on X.

Remark 1.8. [4, Proposition 2] Let $(X,M,N,*,\diamondsuit)$ be an intuitionistic fuzzy metric space, then the triple $(X,M_N,\star)$ is a fuzzy metric on X where M_N is defined on $X \times X \times (0,\infty)$ by $M_N(x,y,t) = 1 - N(x,y,t)$ and $\star$ is the continuous t-norm defined by $a \star b = 1 - [(1-a)\diamondsuit(1-b)]$.

In sequel we simply show the fuzzy metric space $(X,M_N,\star)$ by $(X,N,\star)$.

As a conclusion from above remark and definition of intuitionistic fuzzy metric spaces, we derive the following theorem.

Theorem 1.1. *The 5-tuple $(X,M,N,*,\diamondsuit)$ is an intuitionistic fuzzy metric space if and only if the triples $(X,M,*)$ and $(X,N,\star)$ are fuzzy metric spaces on X and for each $x,y \in X$ and $t \in (0,\infty)$,*

$$M(x,y,t) + N(x,y,t) \leq 1.$$

Remark 1.9. [10] Every fuzzy metric space $(X, M, *)$ is an intuitionistic fuzzy metric space of the form $(X, M, 1-M, *, \lozenge)$ such that t-norm $*$ and t-conorm $\lozenge$ are associated , ie, $x \lozenge y = 1 - [(1-x) * (1-y)]$ for any $x, y \in [0,1]$. We call the intuitionistic fuzzy metric space $(X, M, N, *, \lozenge)$, *the spacial intuitionistic fuzzy metric space* when $M \neq 1 - N$.

Remark 1.10. [11, Proposition.1] Let $(X, M, *)$ be a fuzzy metric space then M is a continuous function on $X \times X \times (0, \infty)$.

By using remarks 1.8 and 1.10 we derive the following:

Theorem 1.2. *Let $(X, M, N, *, \lozenge)$ be an intuitionistic fuzzy metric space, then M and N are continues functions on $X \times X \times (0, \infty)$.*

The followings are examples for intuitionistic fuzzy metric space:

Example 1.11. [10] Let (X, d) be a metric space. Define $a * b = ab$ and $a \lozenge b = min\{1, a+b\}$ for all $a, b \in [0,1]$ and let M_d and N_d be fuzzy sets on $X \times X \times (0, \infty)$ defined as

$$M_d(x,y,t) = \frac{ht^n}{ht^n + md(x,y)} \quad \text{and} \quad N_d(x,y,t) = \frac{md(x,y)}{ht^n + md(x,y)}$$

for all $t, h, m, n \in \mathbb{R}^+$. Then $(X, M_d, N_d, *, \lozenge)$ is an intuitionistic fuzzy metric space and called *induced intuitionistic fuzzy metric space*. The fuzzy metric space $(X, M_d, *)$ is called, *induced fuzzy metric space* [6].

Note the above example holds when t-norm is $a * b = \min\{a, b\}$ and t-conorm is $a \lozenge b = \max\{a, b\}$ and hence the 5-tuple $(X, M, N, *, \lozenge)$ is an intuitionistic fuzzy metric space with respect to any continuous t-norm and continuous t-conorm. In the above example by taking $h = m = n = 1$, we get

$$M_d(x,y,t) = \frac{t}{t + d(x,y)} \quad \text{and} \quad N_d(x,y,t) = \frac{d(x,y)}{t + d(x,y)}$$

The fuzzy metric space $(X, M_d, N_d, *, \lozenge)$ is called, *the standard intuitionistic fuzzy metric space.*

Example 1.12. Let $X = \mathbb{N}$. Define $a * b = \max\{0, a+b-1\}$ and $a \lozenge b = a + b - ab$ for all $a, b \in [0,1]$ and let M and N be fuzzy sets on $X \times X \times (0, \infty)$ as

$$M(x,y,t) = \begin{cases} \frac{x+t}{y+t} & x \leq y \\ \\ \frac{y+t}{x+t} & y \leq x \end{cases} \quad \text{and} \quad N(x,y,t) = \begin{cases} \frac{y-x}{y+t} & x \leq y \\ \\ \frac{x-y}{x+t} & y \leq x \end{cases}$$

for all $x, y \in X$ and $t > 0$, then $(X, M, N, *, \lozenge)$ is an intuitionistic fuzzy metric space, but if we choose t-norm $a * b$ by $\min\{a, b\}$ and t-conorm $a \lozenge b$ by $\max\{a, b\}$, then $(X, M, N, *, \lozenge)$ is not an intuitionistic fuzzy metric space.

Note that, in the above example, t-norm $*$ and t-conorm $\lozenge$ are not associated, and there exists no metric d on X that induces standard intuitionistic fuzzy metric space on X.

In all above examples, M is related to N by $M = 1 - N$. The following is an example of a spacial intuitionistic fuzzy metric space in which $M \neq 1 - N$.

Example 1.13. Let $X = \mathbb{R}^n$ and give d_2 the Euclidean distance on X and d_∞ the max-distance on X.i.e. for each $x = (x_1, x_2, \ldots, x_n)$,$y = (y_1, y_2, \ldots, y_n) \in X$ give

$$d_2(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$$
$$d_\infty(x,y) = \max\{|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n|\}$$

Define $a * b = ab$ and $a \lozenge b = 1 - [(1-a) * (1-b)]$ and let M_{d_2} and N_{d_∞} be fuzzy sets on $X \times X \times (0, \infty)$ as follows

$$M_{d_2}(x,y,t) = \frac{t}{t+d_2(x,y)} \quad \text{and} \quad N_{d_\infty}(x,y,t) = \frac{d_\infty(x,y)}{t+d_\infty(x,y)}$$

for all $x, y \in X$ and $t > 0$, we show that $M_{d_2}(x,y,t) + N_{d_\infty}(x,y,t) \leq 1$, the inequality $d_\infty(x,y) \leq d_2(x,y)$ holds, thus

$$N_{d_\infty}(x,y,t) = \frac{d_\infty(x,y)}{t+d_\infty(x,y)} \leq \frac{d_2(x,y)}{t+d_2(x,y)}$$

consequently

$$M_{d_2}(x,y,t) + N_{d_\infty}(x,y,t) \leq \frac{t}{t+d_2(x,y)} + \frac{d_2(x,y)}{t+d_2(x,y)} = 1.$$

By theorem 1.1 $(X, M_{d_2}, N_{d_\infty}, *, \Diamond)$ is an intuitionistic fuzzy metric space.

2. BEST APPROXIMATION

We begin this section with the concept of t-best approximation points in fuzzy metric spaces introduced by Veeramani [18].
Our reference for best approximation in metric spaces is [16].

Definition 2.1. Let A be a nonempty subset of fuzzy metric space $(X, M, *)$. For $x \in X$ and $t > 0$, define

$$M(A,x,t) = \sup\{M(x,y,t) : y \in A\}$$

An element $y_0 \in A$ is said to be a *t-best approximation point to* x from A if

$$M(y_0,x,t) = M(A,x,t).$$

We denote by $P_A^M(x,t)$ the set of t-best approximation points to x. For $t > 0$ a subset A of a fuzzy metric space $(X, M, *)$ is called t-*proximinal* if for every point $x \in X$, $P_A^M(x,t) \neq \emptyset$.

Example 2.2. [18] Let $X = \mathbb{N}$, define $a * b = ab$ for all $a, b \in [0,1]$, let M be a fuzzy set on $X \times X \times (0,\infty)$ as follows

$$M(x,y,t) = \begin{cases} \frac{x+t}{y+t} & x \leq y \\ \\ \frac{y+t}{x+t} & y \leq x \end{cases}$$

for all $x, y \in X$ and $t > 0$, then $(X, M, *)$ is a fuzzy metric space. Let $A = \{2, 4, 6, \ldots\}$ then we conclude

$$M(A,3,t) = \max\{\tfrac{2+t}{3+t}, \tfrac{3+t}{4+t}\} = \tfrac{3+t}{4+t} = M(3,4,t)$$

Hence, for each $t > 0$, 4 is t-best approximation point to 3 from A. As $M(3,4,t) > M(2,3,t)$, 2 is not a t-best approximation point to 3, so $P_A^M(3,t) = \{4\}$.

A fuzzy metric space $(X, M, *)$ is called strong fuzzy metric space [18, Defination 2.1], if for each x in X and $t > 0$, the map $y \to M(x,y,t)$ is a continuous map on X. Since by remark 1.10 the map $y \to M(x,y,t)$ is always continuous thus every strong fuzzy metric space $(X, M, *)$ is a fuzzy metric space and we can omit the notion of strong fuzzy metric spaces which is used in [18].

Definition 2.3. [18] For $t > 0$, a nonempty subset A of a fuzzy metric space $(X, M, *)$ is said to be t-*approximatively compact* if for each x in X and each sequence y_n in A with $M(y_n,x,t) \to M(A,x,t)$, there exists a subsequence y_{n_k} of y_n converging to an element y_0 in A.

Definition 2.4. [18] For $t > 0$, a nonempty closed subset A of a fuzzy metric space $(X, M, *)$ is said to be *t-boundedly compact* if for each x in X and $0 < r < 1$, the set $B[x,r,t] \cap A$ is a compact subset of X.

Remark 2.5. [18] Let (X, d) be a metric space and $A \subseteq X$, then A is a approximatively compact set in the metric space (X, d) if and only if for any $t > 0$, A is a t-approximatively compact set in the induced fuzzy metric space $(X, M_d, *)$.

Veeramani proved that every nonempty t-approximatively compact subset of a fuzzy metric space is t-proximinal and every t-boundedly compact subset of fuzzy metric space is t-approximatively compact [18, Theorem 2.10,Theorem 2.16 respectively]. Also it can be easily proved that every t-proximinal set is a closed set. Thus each of the following properties in fuzzy metric spaces implies the next one:compact, t-boundedly compact, t-approximatively compact, t-proximinal and closed.

Now we define the notions of t-approximatively compact sets and t-best approximation points in intuitionistic fuzzy metric space$(X, M, N, *, \diamondsuit)$.

Definition 2.6. Let A be a subset of intuitionistic fuzzy metric space $(X, M, N, *, \diamondsuit)$, for $x \in X, t > 0$ let

$$M(A, x, t) = \sup\{M(y, x, t) : y \in A\} \quad \text{and} \quad N(A, x, t) = \inf\{N(y, x, t) : y \in A\}$$

We say A is a t-*proximinal* subset of X (respect to (M, N)) if for each x in X there exist two elements $y_1, y_2 \in A$ such that $M(y_1, x, t) = M(A, x, t)$ and $N(y_2, x, t) = N(A, x, t)$. In this case we say y_1, y_2 are t-*best approximation points* to x (respect to (M, N) from A). We denote by $P_A^{(M,N)}(x, t)$ the set of $\{a \in A; M(a, x, t) = M(A, x, t), N(a, x, t) = N(A, x, t)\}$. If $(X, M, N, *, \diamondsuit)$ be a non-spacial intuitionistic fuzzy metric space, then we have $M = 1 - N$ and if we define $P_A^N(x, t)$ by $\{a \in A; N(a, x, t) = N(A, x, t)\}$then $P_A^M(x, t) = P_A^N(x, t)$ and we can choose $y_1 = y_2$. In this case we say y_1 is a t-*best approximation point* to x (respect to (M, N) from A). Notice by theorem 1.1, $(X, N, \star)$ is a fuzzy metric space and we have $P_A^{(M,N)}(x, t) = P_A^M(x, t) \cap P_A^N(x, t)$.

Next examples illustrate the last definition.

Example 2.7. Take $X = \mathbb{R}^2$, let $(X, M_{d_2}, N_{d_\infty}, *, \diamondsuit)$ be an intuitionistic fuzzy metric space defined in example 1.13 and take $a = (0, 5/4), b = (1, 1) \in X$. Define $A \subseteq X$ by a line that connect a to b. i.e. $A = \{\lambda a + (1 - \lambda)b; \lambda \in [0, 1]\}$. We observe that $d_2(A, x_0) = \inf\{\sqrt{x^2 + y^2}; (x, y) \in A\} = 5/4$ and we find there exists exactly one element $y_1 = (0, 5/4)$ in A such that $d_2(A, x_0) = d_2(y_1, x_0)$. On the other hand, we observe that $d_\infty(A, x_0) = \inf\{\max\{|x|, |y|\}, (x, y) \in A\} = 1$ and we find there exists exactly one element $y_2 = (1, 1)$ in A such that $d_\infty(A, x_0) = d_\infty(y_2, x_0)$, consequently, for every $t > 0$

$$\begin{aligned} M_{d_2}(A, x_0, t) &= \sup\{M_{d_2}(y, x_0, t); y \in A\} = \sup\{\frac{t}{t + d_2(x_0, y)}; y \in A\} \\ &= \frac{t}{t + \inf\{d_2(x_0, y); y \in A\}} = \frac{t}{t + d_2(x_0, y_1)} \end{aligned}$$

Consequently, there exists a unique point $y_1 = (0, 5/4)$ in A such that $M_{d_2}(A, x_0, t) = M_{d_2}(y_1, x_0, t)$ and by similar reasoning we find there exists a unique point $y_2 = (1, 1)$ in A such that $N_{d_\infty}(A, x_0, t) = N_{d_\infty}(y_2, x_0, t)$, so $y_1 = (0, 5/4)$ and $y_2 = (1, 1)$ are t-best approximation points to $x = (0, 0)$ (respect to (M_{d_2}, N_{d_∞})) and $P_A^{(M_{d_2}, N_{d_\infty})}(x, t) = \emptyset$.

Example 2.8. In the example 2.7, if we replace the fuzzy set N_{d_∞} by N_{d_2}, then $y_1 = (0, 5/4)$ is a t-best approximation point to $x = (0, 0)$ respect to (M_{d_2}, N_{d_2}) and $P_A^{(M_{d_2}, N_{d_2})}(x, t) = \{(0, 5/4)\}$.

Remark 2.9. **(a):** For any $t > 0$, A is a t-proximinal subset of X in fuzzy metric space $(X, M, *)$ if and only if A is a t-proximinal subset of X in fuzzy metric space $(X, M, 1-M, *, \Diamond)$. If A is a subset of X, then for each $x \in X$, $P_A^M(x,t) = P_A^{(M,1-M)}(x,t)$.

(b): Suppose $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space and A a subset of X then for every x in X, $y_1, y_2 \in A$ are t-best approximation points to x (respect to (M, N)) in the intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ if and only if y_1 and y_2 are t-best approximation points to x in the fuzzy metric spaces $(X, M, *)$ and $(X, N, \star)$ respectively.

Example 2.10. Consider the intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ in the example 1.12, we have $M = 1 - N$. Let $A = \{2, 4, 6, \ldots\}$, we conclude from the above remark and example 2.2

$$M(A, 3, t) = \max\{\tfrac{2+t}{3+t}, \tfrac{3+t}{4+t}\} = \tfrac{3+t}{4+t} = M(3, 4, t)$$

and

$$N(A, 3, t) = 1 - M(A, 3, t) = 1 - M(3, 4, t) = N(3, 4, t)$$

Hence for each $t > 0$, 4 is t-best approximation point to 3. As $M(3, 4, t) > M(2, 3, t)$, 2 is not a t-best approximation point to 3 and $P_A^M(3,t) = P_A^{(M,N)}(3,t) = \{4\}$.

Remark 2.11. Let (X, d) be a metric space and A a nonempty subset of M, then the following are equivalent.

(a): $y_0 \in A$ is a t-best approximation point to $x \in X$ in the metric space (X, d).

(b): $y_0 \in A$ is a t-best approximation point to $x \in X$ in the induced fuzzy metric space $(X, M_d, *)$.

(c): $y_0 \in A$ is a t-best approximation point to $x \in X$ in the induced intuitionistic fuzzy metric space $(X, M_d, N_d, *,$
$\Diamond)$.

Definition 2.12. For $t > 0$, a nonempty subset A of an intuitionistic fuzzy metric spaces $(X, M, N, *, \Diamond)$ is said to be t-*approximatively compact* if for each x in X and sequences each x_n and y_n in X with $M(y_n, x, t) \to M(A, x, t)$ and $N(x_n, x, t) \to N(A, x, t)$, there exist subsequences x_{n_k} of x_n and y_{n_k} of y_n converging to elements x_0 and $y_0 \in A$, respectively.

Remark 2.13. **(a):** If A be a compact subset of X in intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ then for each $t > 0$, A is a t-approximatively compact set.

(b): A is a approximatively compact subset of X in metric space (X, d) if and only if for each $t > 0$, A is a t-approximatively compact subset of X in the induced intuitionistic fuzzy metric space $(X, M_d, N_d, *, \Diamond)$.

(c): For each $t > 0$, if A is a t-approximatively compact subset of X in intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ then A will be a t-approximatively compact subset of X in fuzzy metric space $(X, M, *)$.

Theorem 2.1. *Let A be a t-approximatively compact subset of X in intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ then A is closed.*

Proof. Let $t > 0$ and A be a t-approximatively compact subset of an intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$, then by remark 2.13, A is a t-approximatively compact set in fuzzy metric space $(X, M, *)$, thus by using [18, Theorem 2.11] A is a closed set in fuzzy metric space $(X, M, *)$ and since the two topologies τ_M

and $\tau_{(M,N)}$ coincides in X, A is a closed set in intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$. □

Theorem 2.2. *For $t > 0$, let A be a nonempty t-approximatively compact subset of X in an intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ then A is a t-proximinal set.*

Proof. For each x in X, we have

$$M(A, x, t) = \sup\{M(y, x, t); y \in A\} \quad \text{and} \quad N(A, x, t) = \inf\{N(y, x, t); y \in A\}$$

consequently there exist sequences x_n and y_n in A such that

$$M(y_n, x, t) \to M(A, x, t) \quad \text{and} \quad N(x_n, x, t) \to N(A, x, t)$$

since A is a t-approximatively compact set, subsequences y_{n_k} of y_n and x_{n_k} of x_n and points $x_0, y_0 \in A$ exist such that $x_{n_k} \to x_0$ and $y_{n_k} \to y_0$. By theorem 1.2, M and N are continuous functions thus we have

$$M(y_{n_k}, x, t) \to M(y_0, x, t) \quad \text{and} \quad N(x_{n_k}, x, t) \to N(x_0, x, t)$$

so we conclude

$$M(y_0, x, t) = M(A, x, t) \quad \text{and} \quad N(x_0, x, t) = N(A, x, t)$$

consequently, y_0, x_0 are t-best approximation points to x from A (respect to (M, N)), i.e. A is a t-proximinal set. □

Definition 2.14. For $t > 0$, a nonempty closed subset A of an intuitionistic fuzzy metric $(X, M, N, *, \Diamond)$ is said to be t-*boundedly compact* if for each x in X and $0 < r < 1$, the set $B_{(M,N)}[x, r, t] \cap A$ is a compact subset of X.

Theorem 2.3. *Let $(X, M, N, *, \Diamond)$ is an intuitionistic fuzzy metric space, If A is a nonempty t-boundedly compact subset of X then A is a t-approximatively compact set.*

Proof. By remark 1.7 we have $B_M[x, r, t] = B_{(M,N)}[x, r, t]$, thus A is a t-boundedly compact set in the fuzzy metric space $(X, M, *)$ if and only if A is a t-boundedly compact set in the intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$. The proof is an immediate consequence of [18, Theorem 2.16]. □

Remark 2.15. Since in an intuitionistic fuzzy metric space a set is compact if and only if it is sequentially compact, thus for each $t > 0$, if A is a t-approximatively compact set then for each x in X the set $P_A^{(M,N)}(x, t)$ is a compact set.

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