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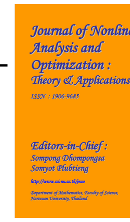

ON THE VECTOR MIXED QUASI-VARIATIONAL INEQUALITY PROBLEMS[◇]

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ABSTRACT. In this paper, we introduce and consider a new class of vector mixed quasi variational inequality and vector complementarity problem in a topological vector space. We show that under certain conditions the solution set of the vector mixed quasi complementarity problem equals to the solution set of the vector mixed quasi variational inequalities. Using the Ky Fan's lemma, we study the existence of a solution of the vector mixed quasi variational inequalities and vector mixed quasi complementarity problems. Moreover we discuss on some of our assumptions. Our results extend those of Farajzadeh et al [Mixed quasi complementarity problems in topological vector spaces, J. Global. Optim.,45 (2009) 229 - 235] to the vector case.

KEYWORDS : Complementarity problems; Mixed quasi-variational inequality; Existence results.

1. INTRODUCTION

Complementarity problems theory, which was introduced and studied by Lemke [14]and Cottle and Dantzig [5] in early 1960's, has enjoyed a vigorous and dynamics growth. Complementarity problems have been extended and generalized in various directions to study a large class of problems arising in industry, finance, optimization, regional, physical, mathematical and engineering sciences, see [1-19]. Equally important is the mathematical subject known as variational inequalities which was introduced in early 1960's. For the applications, physical formulation, numerical

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methods, dynamical system and sensitivity analysis of the mixed quasi variational inequalities, see [1,5,15] and the references therein. It has been shown that if the set involved in complementarity problems and variational inequalities is a convex cone together with suitable assumptions then both the complementarity problems and variational inequalities are equivalent, see Karamardian [13]. This equivalence has played a central and crucial role in suggesting new and unified algorithms for solving complementarity problem and its various generalizations and extensions, see [1-19] and the references therein for more details.

Inspired and motivated by the reference [9], we introduce and analyze a new class of vector mixed quasi variational inequality and vector mixed complementarity problem in topological vector spaces. These classes are quite general and unifies several classes of complementarity problems in a general framework. Under suitable conditions, we establish the equivalence between the vector mixed quasi complementarity problem and vector mixed quasi variational inequalities problem. This alternative equivalence is used to discuss several existence results for the solution of the these problems by using Fan's lemma in topological vector spaces. The vector mixed quasi variational inequalities include vector mixed quasi complementarity problems, f-complementarity problems, general complementarity problems, various classes of vector variational inequalities and related vector optimization problems as special cases.

2. PRELIMINARIES

Let X and Y be two real Hausdorff topological vector spaces and K be a nonempty subset of X. Denote by $L(X,Y)$ the space of all continuous linear mappings from X into Y, and $\langle t,x\rangle$ be the value of the linear continuous mapping $t\in L(X,Y)$ at x. Suppose that $C : K \longrightarrow 2^Y$ is a set valued map with nonempty pointed (that is, $C(u)\cap -C(u)=\{0\}$ for all $u\in K$) convex cone values, $F : K\times K \longrightarrow Y$, and $T : K \longrightarrow L(X,Y)$.

We consider the problem of finding $u\in K$ such that

$$\langle Tu,u\rangle + F(u,u) = 0, \quad \langle Tu,v\rangle + F(v,u)\in C(u), \quad \forall v\in K, \tag{2.1}$$

which we call it the *vector mixed quasi complementarity problem (VMQCP)*.

We note that if $Y=\Re$ (real numbers) and $C(u)=[0,\infty),\ F(v,u)=f(v), \forall u,v\in K$, then problem (1) is equivalent to finding $u\in K$ such that

$$\langle Tu,u\rangle + f(u) = 0, \quad \langle Tu,v\rangle + f(v)\geq 0, \quad \forall u,v\in K, \tag{2.2}$$

which is known as the f-complementarity problem, introduced and studied by Itoh et al [12]. Moreover if $F(v,u)=f(v), \quad \forall v\in K$, problem (1) reduces to the vector version of Itoh et al's problem introduced in [12]. For the applications and numerical methods of problem (2), see [16, 2, 10].

If $F(u,v)=0$, for all $u,v\in K$, and $K^*\equiv\{u\in X^* : \langle u,v\rangle\geq 0, \quad \forall v\in K\}$ is a polar (dual) cone of the convex cone K, then the mixed quasi complementarity problem (VMQCP) is equivalent to finding $u\in K$ such that

$$Tu\in K^* \quad \text{and} \quad \langle Tu,u\rangle = 0, \tag{2.3}$$

which is called the general complementarity problem. For the recent applications, numerical results and formulation of the complementarity problems, see [2,3,4,7,9,16] and the references therein.

Related to the vector mixed quasi complementarity problem (1), we consider the problem of finding $u \in K$, where K is a nonempty subset of X, such that

$$\langle Tu, v-u\rangle + F(v,u) - F(u,u) \in C(u), \quad \forall v \in K, \tag{2.4}$$

which we call it the *vector mixed quasi variational inequality* (VMQVIP). For the formulation, numerical results, existence results, sensitivity analysis and dynamical aspects of the scalar mixed quasi variational inequalities (that is $Y = \Re$), see [1,8,10,13-19] and the references therein.

It is obvious that any solution of (VMQCP) is a solution of (VMQVIP). The following example shows that the converse does not hold in general.

Example 2.1. Let $X = Y = \Re,\ K = [0, +\infty),\ F(x,y) = 1$, for all $x, y \in K,\ C(x) = [0, +\infty)$ for all $x \in K$ and define $T : K \longrightarrow \Re^* = \Re$ by

$$T(x) = \begin{cases} 0 & \text{if} \quad x = 0, \\ -1 & \text{otherwise.} \end{cases}$$

Then $u = 0$ is a solution of (VMQVIP), whereas (VMQCP) hasn't any solution. We now show that the problems (1) and (4),under some conditions, are equivalent, that is their solution sets are equal and this is the main motivation of our next result.

Theorem 2.2. *Let K be a nonempty subset of a topological vector space X with $2K \subseteq K$, and $0 \in K$. If $F(2v,u) = 2F(v,u)$, for all $u, v \in K$, then the solution sets of (VMQCP) and (VMVIP) are equal.*

Proof. It suffices to show that any solution of (VMQVIP) is a solution of (VMQCP). Let $u \in K$ be a solution of the vector mixed quasi variational inequality (4). Then by taking $v = 0$ and $v = 2u$ in (4), we have

$$\langle Tu, -u\rangle + F(0,u) - F(u,u) \in C(u),$$
$$\langle Tu, u\rangle + F(2u,u) - F(u,u) \in C(u),$$

which implies, using $F(0,u) = 0,\ F(2u,u) = 2F(u,u)$, and $C(u) \cap -C(u) = \{0\}$, that

$$\langle Tu, u\rangle + F(u,u) = 0. \tag{2.5}$$

Also, from (5) and (4), we have , for all $v \in K$,

$$\begin{aligned} \langle Tu, v\rangle + F(v,u) &= \\ \langle Tu, v\rangle + F(v,u) - (\langle Tu, u\rangle + F(u,u)) &= \\ \langle Tu, v-u\rangle + F(v,u) - F(u,u) &\in C(u) \end{aligned}$$

that is,

$$\langle Tu, v\rangle + F(v,u) \in C(u), \quad \forall v \in K. \tag{2.6}$$

This shows that $u \in K$ is a solution of (VMQCP). □

Remark 2.3. (a) If K is a closed convex cone, then $0 \in K$, and $2K \subseteq K$, but every convex set with $0 \in K$, does not so. For instance, $K = N \cup \{0\}$, the set N denotes the natural numbers, is not a convex cone while is a nonempty set with $2K \subseteq K$, and $0 \in K$.

(b) If F is positively homogeneous in the first variable then $F(2u,v) = 2F(u,v),\ \forall u, v \in K$. However the converse is not true, for instance, $F(u,v) = 0,$ for u rational and $F(u,v) = u,$ for u irrational, which is not positively homogeneous but satisfies $F(2u,v) = 2F(u,v)$ for $u, v \in K$.

In the rest of this section, we recall some definitions and Ky Fan's lemma, which will be used in the next section.
We shall denote by 2^A the family of all subsets of A and by $\mathcal{F}(A)$ the family of all nonempty finite subsets of A. Let X be a nonempty set, Y a topological space, and $\Gamma : X \longrightarrow 2^Y$ a multi-valued map. Then, Γ is called transfer closed-valued if, for every $y \notin \Gamma(x)$, there exists $x' \in X$ such that $y \notin cl\Gamma(x')$, where cl denotes the closure of a set. It is well-known that, $\Gamma : X \longrightarrow 2^Y$ is transfer closed-valued if and only if

$$\bigcap_{x\in X} \Gamma(x) = \bigcap_{x\in X} cl\Gamma(x).$$

If $B \subseteq Y$ and $A \subseteq X$, then $\Gamma : A \longrightarrow 2^B$ is called transfer closed-valued if the set valued mapping $x \longrightarrow \Gamma(x) \cap B$ is transfer closed-valued. In this case where $X = Y$ and $A = B$, Γ is called transfer closed-valued on A.
Let X and Y are two topological vector space, K be a nonempty subset of Y, and $C \subseteq Y$ be nonempty and convex. The map $f : K \longrightarrow Y$ is said to be C-convex if for each $0 \leq \lambda \leq 1$ and $x_1, x_2 \in K$ we have

$$\lambda f(x_1) + (1-\lambda)f(x_2) - f(\lambda x_1 + (1-\lambda)x_2) \in C.$$

Let K be a nonempty convex subset of a topological vector space X and let K_0 be a subset of K. A set valued map $\Gamma : K_0 \longrightarrow 2^K$ is said to be a *KKM map* when

$$coA \subseteq \bigcup_{x\in A} \Gamma(x), \quad \forall\ A\ \in \mathcal{F}(K_0),$$

where co denotes the convex hull.

Lemma 2.4. *(Ky Fan [7]). Let K be a nonempty subset of a topological vector space X and $F : K \longrightarrow 2^X$ be a KKM mapping with closed values. Assume that there exist a nonempty compact convex subset B of K such that $D = \bigcap_{x\in B} F(x)$ is compact. Then*

$$\bigcap_{x\in K} F(x) \neq \emptyset.$$

3. MAIN RESULTS

In this section, we provide some existence theorems in order to guarantee the solution set of (VMQVIP) and (VMQCP) be nonempty and relatively compact.
Throughout this section ,unless otherwise specified, let X and Y be two real Hausdorff topological vector spaces and K be a nonempty convex subset of X. Denote by $L(X,Y)$ the space of all continuous linear mappings from X into Y, and $\langle t, x\rangle$ be the value of the linear continuous mapping $t \in L(X,Y)$ at x. Suppose that $C : K \longrightarrow 2^Y$ is a set valued map with nonempty convex cone values and $F : K \times K \longrightarrow Y$, $T : K \longrightarrow L(X,Y)$ are two mappings.

We need the following lemma for the next result.

Lemma 3.1. *Let X be a topological vector space and $E \subseteq X$ be compact and convex. Let $A = \{a_1, ..., a_n\}$ be a finite subset of X. Then $co(A \cup E)$ is compact.*

Proof. Let $\Delta_n = \{\Sigma_{i=1}^n \lambda_i e_i \ : \ \lambda_i \geq 0, \ \Sigma_{i=1}^n \lambda_i = 1\}$, where (e_i) is the standard base of $\mathbb{R}^n$. Define $\Lambda : \Delta_{n+1} \times \{a_1\} \times ... \times \{a_n\} \times E \longrightarrow co(A \cup E)$ by

$$\Lambda(\lambda_1, ..., \lambda_{n+1}, a_1, ..., a_n, e) = \Sigma_{i=1}^n \lambda_i a_i + \lambda_{n+1} e.$$

It is clear that Λ is an onto continuous mapping and $\Delta_{n+1} \times \{a_1\} \times ... \times \{a_n\} \times E$ is compact. Then $co(A \cup E) = \Lambda(\Delta_{n+1} \times \{a_1\} \times ... \times \{a_n\} \times E)$ is compact. The proof is complete. □

Theorem 3.2. *Assume that*

(a) *there exist a nonempty compact subset B and a nonempty convex compact subset D of K such that*

(i) for each $x \in K \backslash B$, there exists $y \in D$ such that $\langle T(x), y - x\rangle + F(y, x) - F(x, x) \notin C(x)$;

(ii) for each fixed $y \in K$, the set valued mapping $G : K \longrightarrow 2^Y$ defined by

$$G(y) = \{x \in K : \langle Tx, y - x\rangle + F(y, x) - F(x, x) \in C(x)\}$$

is transfer closed on each compact convex subset K containing D

(b) *there exists a mapping $h : K \times K \longrightarrow Y$ such that*
(i) $h(x, x) \in C(x), \ \forall \ x \in K$;
(ii) $\langle T(x), y - x\rangle + F(y, x) - F(x, x) - h(x, y) \in C(x), \ \forall x, y \in K, ;$
(iii) the set $\{y \in K : h(x, y) \notin C(x)\}$ is convex, $\forall \ x \in K$.

Then (VMQVI) has a solution. Moreover, the solution set of (VMQVI) is relatively compact.

Proof. For each fixed $A \in \mathcal{F}(K)$, define the multi-valued maps $\Gamma_A, \widehat{\Gamma_A} : co(A \cup D) \longrightarrow 2^{co(A \cup D)}$ as follows:

$$\Gamma_A(y) = \{x \in co(A \cup D) : \langle T(x), y - x\rangle + F(y, x) - F(x, x) \in C(x)\},$$
$$\widehat{\Gamma_A}(y) = \{x \in co(A \cup D) : h(x, y) \in C(x)\}.$$

We show that $\widehat{\Gamma}$ is a KKM mapping. On the contrary, suppose there exists $M = \{x_1, x_2, ..., x_n\} \subseteq co(A \cup D)$, and $z \in coM$, such that $z \notin \cup_{i \in \{1,2,...,n\}} \widehat{\Gamma}(x_i)$. Then $h(z, x_i) \notin C(z)$ for $i = 1, 2, 3, ..., n$. It follows by (b)(iii) that, $h(z, z) \notin C(z)$ contradicting (b)(i). Hence $\widehat{\Gamma_A}$ is a KKM map and so Γ_A is a KKM map (since $\widehat{\Gamma}(y) \subseteq \Gamma(y)$, for all $y \in K$). By Lemma 2.1 the set $co(A \cup D)$ is compact and convex and hence by Lemma 1.4 the intersection $\bigcap_{x \in co(A \cup D)} cl\Gamma_A(x)$ is nonempty and so by the assumption (ii) of (a) we have

$$\bigcap_{x \in co(A \cup D)} cl\Gamma_A(x) = \bigcap_{x \in co(A \cup D)} \Gamma_A(x).$$

Hence $M_A = \bigcap_{x \in co(A \cup D)} \Gamma_A(x)$ is nonempty. We consider the family

$$\sum = \{M_A : A \in \mathcal{F}(K)\}.$$

It is clear that $\sum$ has finite intersection property (note if $A_1, ..., A_n \in \mathcal{F}(K)$ then $\cap_{i=1}^n M_{A_i} \supseteq M_{\cup_{i=1}^n A_i}$). Hence $\bigcap_{A \in \mathcal{F}(K)} M_A$ is nonempty (note (a)(i) implies $M_A \subseteq B$ for each $A \in \mathcal{F}(K)$ and the family $\sum$ has finite intersection property) and so there exists $\overline{x} \in \bigcap_A clM_A$. Now if x is an arbitrary element of K then we claim that

$$\langle T(\overline{x}), x - \overline{x}\rangle + F(x, \overline{x}) - F(\overline{x}, \overline{x}) \in C(\overline{x}),$$

that is $\overline{x}$ is a solution of (VMQVI).

To see this let $S = \{x, \overline{x}\}$. Then

$$\overline{x} \in (\bigcap_{A \in \mathcal{F}(K)} cl_K M_A) \cap co(D \cup S) \subseteq \bigcap_{y \in S} cl_{co(D\cup S)} \Gamma_S(y) \cap co(D \cup S) =$$

$$\bigcap_{y \in S} \Gamma_S(y) \cap co(D \cup S) \subseteq \Gamma_S(x),$$

and so this proves the assertion. It is obvious from (a)(i) that the solution set of (VMQVI) is a subset of B. □

Remark 3.3. Condition (a)(ii) of Theorem 2.2 holds when T, F are continuous mappings and the graph of the mapping C is closed. The following simple example shows that the continuity of the maps isn't necessary condition to hold (a)(ii). Let $X = Y = \Re$, and K be a non-singleton convex subset of X and let $T(x) = 0, F(x, x) = 0, F(x, y) = 1$ for $x \neq y,$. Then F is not continuous but the condition (a)(ii) is satisfied. can fail.

Corollary 3.4. *Assume that:*

(a) *there exist a nonempty compact subset B and a nonempty convex compact subset D of K such that*

(i) for each $x \in K \backslash B$, there exists $y \in D$ such that $\langle T(x), y - x\rangle + F(y, x) - F(x, x) \notin C(x)$;

(ii) for each fixed $y \in K$, the set valued mapping $G : K \longrightarrow 2^Y$ defined by

$$G(y) = \{x \in K : \langle Tx, y - x\rangle + F(y, x) - F(x, x) \in C(x)\}$$

is transfer closed on each compact convex subset K containing D

(b) the set $\{y \in K : \langle Tx, y - x\rangle + F(y, x) - F(x, x) \notin C(x)\}$ is convex, $\forall\, x \in K$.

Then, (VMQVI) has a solution. Moreover, the solution set of (VMQVI) is relatively compact.

Proof. The result follows from Theorem 2.2 by defining $h(x, y) = \langle Tx, y - x\rangle + F(y, x) - F(x, x)$, for each $x, y \in K$.

Remark 3.5. The condition (b) of Corollary 2.4 holds if the function $y \longrightarrow F(x, y)$ is $C(x)-$ convex for each $x \in K$. To see this let $\lambda \in [0, 1]$, and

$$\langle Tx, y_i - x\rangle + F(y_i, x) - F(x, x) \in Y \backslash C(x), \text{ for } i = 1, 2.$$

Since $Y \backslash C(x)$ is an open set then there exists a balanced neighborhood V of zero such that we have

$$V + (\langle Tx, y_1 - x\rangle + F(y, x) - F(x, x)) \in Y \backslash C(x).$$

Moreover, note $Y \backslash C(x)$ is a cone, for each positive integer n we get

$$\frac{\lambda}{n}(V + (\langle Tx, y_1 - x\rangle + F(y_1, x) - F(x, x)) \in Y \backslash C(x). \qquad (2.1)$$

For sufficiently large positive integer n we have

$$\frac{1-\lambda}{\lambda n}(\langle Tx, y_2 - x\rangle + F(y_2, x) - F(x, x)) \in V \qquad (2.2)$$

From (2.1) and (2.2) we get

$$\lambda(\langle Tx, y_1 - x\rangle + F(y_1, x) - F(x, x)) + (1-\lambda)(\langle Tx, y_2 - x\rangle + F(y_2, x) - F(x, x)) \in Y\backslash C(x) \qquad (2.3).$$

From the $C(x)-$ convexity we have

$$\lambda F(y_1, x) + (1-\lambda)F(y_2, x) - F(\lambda y_1 + (1-\lambda)y_2, x) \in C(x) \ (2.4).$$

Finally (2.3), (2.4) and $(Y\backslash C(x)) - C(x) \subseteq Y\backslash C(x)$ imply the result.

By combining Corollary 2.4 and Remark 2.5 we obtain the following result.

Theorem 3.6. *Assume that:*

(a) *there exist a nonempty compact subset B and a nonempty convex compact subset D of K such that*

(i) for each $x \in K\backslash B$, there exists $y \in D$ such that $\langle T(x), y - x\rangle + F(y, x) - F(x, x) \notin C(x)$;

(ii) for each fixed $y \in K$, the set valued mapping $G : K \longrightarrow 2^Y$ defined by

$$G(y) = \{x \in K : \langle Tx, y - x\rangle + F(y, x) - F(x, x) \in C(x)\}$$

is transfer closed on each compact convex subset K containing D

(b) *the function $y \longrightarrow F(x, y)$ is $C(x)-$ convex, $\forall x \in K$;*

Then, (VMQVI) has a solution. Moreover, the solution set of (VMQVI) is compact.

Theorem 3.7. *Suppose that:*

(a) *there exist a nonempty compact subset B and a nonempty convex compact subset D of K such that*

(i) for each $x \in K\backslash B$, there exist $y \in D$ and an open neighborhood U_x of x in K such that

$$\langle T(z), y - z\rangle + F(y, z) - F(z, z) \notin C(z), \forall z \in U_x;$$

(ii) for each fixed $y \in K$, the set valued mapping $G : K \longrightarrow 2^Y$ defined by

$$G(y) = \{x \in K : \langle Tx, y - x\rangle + F(y, x) - F(x, x) \in C(x)\}$$

is transfer closed on each compact convex subset K containing D

(b) *the function $y \longrightarrow F(x, y)$ is $C(x)-$ convex, $\forall x \in K$;*

Then, (VMQVI) has a solution. Moreover, the solution set of (VMQVI) is nonempty and relative compact in K.

Proof. We define $\Gamma : K \longrightarrow 2^K$ by

$$\Gamma(y) = \{x \in K : \langle T(x), y - x\rangle + F(y, x) - F(x, x) \in C(x)\}.$$

One can see, by using (b) and Remark 2.5, Γ is a KKM map and so $cl\Gamma$ is a KKM mapping (Note $\Gamma(x) \subseteq cl\Gamma(x)$ for all $x \in K$). From (a)(i) we conclude $\bigcap_{x\in D} cl\Gamma(x) \subseteq B$ and hence Lemma 1.4 implies $\bigcap_{x\in D} cl\Gamma(x) \neq \emptyset$. Then for each nonempty finite subset A of K, by Lemmas 2.1, 1.4, we get $\bigcap_{x\in co(D\cup A)} cl\Gamma(x) \neq \emptyset$. Now we can conclude the proof as the same manner of the proof of Theorem 2.2. □

Theorem 3.8. *Suppose that all assumptions of one of the Theorems 2.2, 2.6 or 2.7 or Corollary 2.4 are satisfied. If,* $0 \in K$ *and* $F(2u, v) = 2F(u, v), \forall u, v \in K$*, then, (VMQCP) has a solution. Moreover, the solution set of (VMQCP) is relative compact.*

Proof. The result follows from Theorems 2.2 and 2.3. □

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