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A STRUCTURE THEOREM ON NON-HOMOGENEOUS LINEAR EQUATIONS IN HILBERT SPACES

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ABSTRACT. A very particular by-product of the result announced in the title reads as follows: Let $(X, \langle \cdot, \cdot \rangle)$ be a real Hilbert space, $T : X \longrightarrow X$ a compact and symmetric linear operator, and $z \in X$ such that the equation $T(x) - \|T\|x = z$ has no solution in X. For each $r > 0$, set $\gamma(r) = \sup_{x \in S_r} J(x)$, where $J(x) = \langle T(x) - 2z, x \rangle$ and $S_r = \{x \in X : \|x\|^2 = r\}$. Then, the function γ is C^1, increasing and strictly concave in $]0, +\infty[$, with $\gamma'(]0, +\infty[) =]\|T\|, +\infty[$; moreover, for each $r > 0$, the problem of maximizing J over S_r is well-posed, and one has

$$T(\hat{x}_r) - \gamma'(r)\hat{x}_r = z$$

where $\hat{x}_r$ is the only global maximum of $J_{|S_r}$.

KEYWORDS : Linear equation; Hilbert space; Eigenvalue; Well-posedness.

1. INTRODUCTION AND PRELIMINARIES

Here and in the sequel, $(X, \langle \cdot, \cdot \rangle)$ is real Hilbert space. For each $r > 0$, set

$$S_r = \{x \in X : \|x\|^2 = r\} .$$

In [1], we established the following result (with the usual conventions $\sup \emptyset = -\infty$, $\inf \emptyset = +\infty$):

Theorem A ([1], Theorem 1). *Let $J : X \longrightarrow \mathbf{R}$ be a sequentially weakly continuous C^1 functional, with $J(0) = 0$. Set*

$$\rho = \limsup_{\|x\| \longrightarrow +\infty} \frac{J(x)}{\|x\|^2}$$

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and

$$\sigma = \sup_{x \in X \setminus \{0\}} \frac{J(x)}{\|x\|^2} \; .$$

Let a, b *satisfy*

$$\max\{0, \rho\} \leq a < b \leq \sigma \; .$$

Assume that, for each $\lambda \in]a, b[$*, the functional* $x \longrightarrow \lambda\|x\|^2 - J(x)$ *has a unique global minimum, say* $\hat{y}_\lambda$*. Let* M_a *(resp.* M_b *if* $b < +\infty$ *or* $M_b = \emptyset$ *if* $b = +\infty$*) be the set of all global minima of the functional* $x \longrightarrow a\|x\|^2 - J(x)$ *(resp.* $x \longrightarrow b\|x\|^2 - J(x)$ *if* $b < +\infty$*). Set*

$$\alpha = \max\left\{0, \sup_{x \in M_b} \|x\|^2\right\} \; ,$$

$$\beta = \inf_{x \in M_a} \|x\|^2$$

and, for each $r > 0$,

$$\gamma(r) = \sup_{x \in S_r} J(x).$$

Finally, assume that J *has no local maximum with norm less than* β.

Then, the following assertions hold:

(a_1) *the function* $\lambda \longrightarrow g(\lambda) := \|\hat{y}_\lambda\|^2$ *is decreasing in* $]a, b[$ *and its range is* $]\alpha, \beta[$ *;*

(a_2) *for each* $r \in]\alpha, \beta[$*, the point* $\hat{x}_r := \hat{y}_{g^{-1}(r)}$ *is the unique global maximum of* $J_{|S_r}$ *and every maximizing sequence for* $J_{|S_r}$ *converges to* $\hat{x}_r$ *;*

(a_3) *the function* $r \longrightarrow \hat{x}_r$ *is continuous in* $]\alpha, \beta[$ *;*

(a_4) *the function* γ *is* C^1*, increasing and strictly concave in* $]\alpha, \beta[$ *;*

(a_5) *one has*

$$J'(\hat{x}_r) = 2\gamma'(r)\hat{x}_r$$

for all $r \in]\alpha, \beta[$ *;*

(a_6) *one has*

$$\gamma'(r) = g^{-1}(r)$$

for all $r \in]\alpha, \beta[$.

We want to remark that, in the original statement of [1], one assumes that X is infinite-dimensional and that J has no local maxima in $X \setminus \{0\}$. These assumptions come from [2] whose results are applied to get (a_3), (a_4) and (a_5). The validity of the current formulation just comes from the proofs themselves given in [2] (see also [3]).

The aim of this very short paper is to show the impact of Theorem A in the theory of non-homogeneous linear equations in X.

2. MAIN RESULTS

Throughout the sequel, z is a non-zero point of X and $T : X \longrightarrow X$ is a continuous linear operator.

We are interested in the study of the equation

$$T(x) - \lambda x = z$$

for $\lambda > \|T\|$. In this case, by the contraction mapping theorem, the equation has a unique non-zero solution, say $\hat{v}_\lambda$. Our structure result just concerns such solutions.

As usual, we say that:

- T is compact if, for each bounded set $A \subset X$, the set $\overline{T(A)}$ is compact ;

- T is symmetric if

$$\langle T(x), u\rangle = \langle T(u), x\rangle$$

for all $x, u \in X$.

We also denote by V the set (possibly empty) of all solutions of the equation

$$T(x) - \|T\|x = z$$

and set

$$\theta = \inf_{x\in V} \|x\|^2 \ .$$

Of course, $\theta > 0$. Our result reads as follows:

Theorem 1. - *Assume that T is compact and symmetric .*

For each $\lambda > \|T\|$ and $r > 0$, set

$$g(\lambda) = \|\hat{v}_\lambda\|^2$$

and

$$\gamma(r) = \sup_{x\in S_r} J(x)$$

where

$$J(x) = \langle T(x) - 2z, x\rangle \ .$$

Then, the following assertions hold:

(b_1) *the function g is decreasing in $]\|T\|, +\infty[$ and*

$$g(]\|T\|, +\infty[) =]0, \theta[\ ;$$

(b_2) *for each $r \in]0, \theta[$, the point $\hat{x}_r := \hat{v}_{g^{-1}(r)}$ is the unique global maximum of $J_{|S_r}$ and every maximizing sequence for $J_{|S_r}$ converges to $\hat{x}_r$;*

(b_3) *the function $r \longrightarrow \hat{x}_r$ is continuous in $]0, \theta[$;*

(b_4) *the function γ is C^1, increasing and strictly concave in $]0, \theta[$;*

(b_5) *one has*

$$T(\hat{x}_r) - \gamma'(r)\hat{x}_r = z$$

for all $r \in]0, \theta[$;

(b_6) *one has*

$$\gamma'(r) = g^{-1}(r)$$

for all $r \in]0, \theta[$.

Before giving the proof of Theorem 1, we establish the following

Proposition 1. - *Let T be symmetric and let J be defined as in Theorem 1. Then, for $\tilde{x} \in X$, the following are equivalent:*

(i) *$\tilde{x}$ is a local maximum of J .*

(ii) *$\tilde{x}$ is a global maximum of J .*

(iii) *$T(\tilde{x}) = z$ and $\sup_{x\in X}\langle T(x), x\rangle \le 0$.*

Proof. First, observe that, since T is symmetric, the functional J is Gâteaux differentiable and its derivative, J', is given by

$$J'(x) = 2(T(x) - z)$$

for all $x \in X$ ([4], p. 235). By the symmetry of T again, it is easy to check that, for each $x \in X$, the inequality

$$J(\tilde{x} + x) \le J(\tilde{x}) \tag{1}$$

is equivalent to

$$\langle 2(T(\tilde{x}) - z) + T(x), x\rangle \le 0 \ . \tag{2}$$

Now, if (i) holds, then $J'(\tilde{x}) = 0$ (that is $T(\tilde{x}) = z$) and there is $\rho > 0$ such that (1) holds for all $x \in X$ with $\|x\| \le \rho$. So, from (2), we have $\langle T(x), x\rangle \le 0$ for the same

x and then, by linearity, for all $x \in X$, getting (iii). Vice versa, if (iii) holds, then (2) is satisfied for all $x \in X$ and so, by (1), $\tilde{x}$ is a global maximum of J, and the proof is complete. △

Proof of Theorem 1. For each $x \in X$, we clearly have

$$J(x) \leq \|T(x) - 2z\|\|x\| \leq \|T\|\|x\|^2 + 2\|z\|\|x\|$$

and so

$$\limsup_{\|x\| \longrightarrow +\infty} \frac{J(x)}{\|x\|^2} \leq \|T\| \ . \tag{3}$$

Moreover, if $v \in X \setminus \{0\}$ and $\mu \in \mathbf{R} \setminus \{0\}$, we have

$$\frac{J(\mu v)}{\|\mu v\|^2} \geq -2\frac{\langle z, v\rangle}{\mu\|v\|^2} - \|T\|$$

and so

$$\limsup_{x \longrightarrow 0} \frac{J(x)}{\|x\|^2} = +\infty \ . \tag{4}$$

Moreover, the compactness of T implies that J is sequentially weakly continuous ([4], Corollary 41.9). Now, let $\lambda \geq \|T\|$. For each $x \in X$, set

$$\Phi(x) = \|x\|^2 \ .$$

Then, for each $x, v \in X$, we have

$$\langle \lambda\Phi'(x) - J'(x) - (\lambda\Phi'(v) - J'(v)), x - v\rangle = \langle 2\lambda(x - v) - 2(T(x) - T(v)), x - v\rangle \geq$$
$$2\lambda\|x - v\|^2 - 2\|T(x) - T(v)\|\|x - v\| \geq 2(\lambda - \|T\|)\|x - v\|^2 \ . \tag{5}$$

From (5) we infer that the derivative of the functional $\lambda\Phi - J$ is monotone, and so the functional is convex. As a consequence, the critical points of $\lambda\Phi - J$ are exactly its global minima. So, $\hat{v}_\lambda$ is the only global minimum of $\lambda\Phi - J$ if $\lambda > \|T\|$ and V is the set of all global minima of $\|T\|\Phi - J$. Now, assume that J has a local maximum, say w. Then, by Proposition 1, w is a global minimum of $-J$ and $\sup_{x \in X}\langle T(x), x\rangle \leq 0$. Since T is symmetric, this implies, in particular, that $\|T\|$ is not in the spectrum of T. So, V is a singleton. By Proposition 1 of [1], we have

$$\|w\|^2 \geq \theta \ .$$

In other words, J has no local maximum with norm less than θ. At this point, taking (3) and (4) into account, we see that the assumptions of Theorem A are satisfied (with $a = \|T\|$ and $b = +\infty$, and so $\alpha = 0$ and $\beta = \theta$), and the conclusion follows directly from that result. △

Some remarks on Theorem 1 are now in order.

Remark 1. - Each of the two properties assumed on T cannot be dropped. Indeed, consider the following two counter-examples.

Take $X = \mathbf{R}^2$, $z = (1, 0)$ and $T(t, s) = (t + s, s - t)$ for all $(t, s) \in \mathbf{R}^2$. So, T is compact but not symmetric. In this case, we have

$$\hat{x}_r = (-\sqrt{r}, 0) \ ,$$
$$\gamma(r) = r + 2\sqrt{r}$$

for all $r > 0$. Hence, in particular, we have

$$T(\hat{x}_r) - \gamma'(r)\hat{x}_r = (1, \sqrt{r}) \neq z \ .$$

That is, (b_5) is not satisfied.

Now, take $X = l_2$, $z = \{w_n\}$, where $w_2 = 1$ and $w_n = 0$ for all $n \neq 2$, and $T(\{x_n\}) = \{v_n\}$ for all $\{x_n\} \in l_2$, where $v_1 = 0$ and $v_n = x_n$ for all $n \geq 2$.

So, T is symmetric but not compact. In this case, we have $\theta = +\infty$ and

$$\gamma(r) = r - 2\sqrt{r}$$

for all $r \geq 4$. Hence, γ is not strictly concave in $]0, +\infty[$.

Remark 2. - Note that the compactness of T serves only to guarantee that the functional $x \longrightarrow \langle T(x), x\rangle$ is sequentially weakly continuous. So, Theorem 1 actually holds under such a weaker condition.

Remark 3. - A natural question is: if assertions $(b_1) - (b_6)$ hold, must the operator T be symmetric and the functional $x \longrightarrow \langle T(x), x\rangle$ sequentially weakly continuous ?

Remark 4. - Note that if T, besides to be compact and symmetric, is also positive (i.e. $\inf_{x\in X}\langle T(x), x\rangle \geq 0$), then, by classical results, the operator $x \longrightarrow T(x) - \|T\|x$ is not surjective, and so there are $z \in X$ for which the conclusion of Theorem 1 holds with $\theta = +\infty$.

We conclude with an application of Theorem 1 to a classical Dirichlet problem.

So, let $\Omega \subset \mathbf{R}^n$ be a bounded domain with smooth boundary. Let λ_1 be the first eigenvalue of the problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Fix a non-zero continuous function $\varphi : \overline{\Omega} \longrightarrow \mathbf{R}$.

For each $\mu \in]0, \lambda_1[$, let u_μ be the unique classical solution of the problem

$$\begin{cases} -\Delta u = \mu(u + \varphi(x)) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Also, set

$$\psi(\mu) = \int_\Omega |\nabla u_\mu(x)|^2 dx$$

and

$$\eta(r) = \sup_{u\in U_r} \Phi(u)$$

where

$$\Phi(u) = \int_\Omega |u(x)|^2 dx + 2\int_\Omega \varphi(x)u(x)dx$$

and

$$U_r = \left\{ u \in H^1_0(\Omega) : \int_\Omega |\nabla u(x)|^2 dx = r \right\} .$$

Finally, denote by A the set of all classical solutions of the problem

$$\begin{cases} -\Delta u = \lambda_1(u + \varphi(x)) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

and set

$$\delta = \inf_{u\in A} \int_\Omega |\nabla u(x)|^2 dx .$$

Then, by using standard variational methods, we can directly draw the following result from Theorem 1 :

Theorem 2. - *The following assertions hold:*
(c_1) *the function* ψ *is increasing in* $]0, \lambda_1[$ *and one has*

$$\psi(]0, \lambda_1[) =]0, \delta[;$$

(c_2) *for each* $r \in]0, \delta[$, *the function* $w_r := u_{\psi^{-1}(r)}$ *is the unique global maximum of* $\Phi_{|U_r}$ *and each maximizing sequence for* $\Phi_{|U_r}$ *converges to* w_r *with respect to the topology of* $H_0^1(\Omega)$ *;*
(c_3) *the function* $r \longrightarrow w_r$ *is continuous in* $]0, \delta[$ *with respect to the topology of* $H_0^1(\Omega)$ *;*
(c_4) *the function* η *is* C^1, *increasing and strictly concave in* $]0, \delta[$ *;*
(c_5) *for each* $r \in]0, \delta[$, *the function* w_r *is the unique classical solution of the problem*

$$\begin{cases} -\Delta u = \frac{1}{\eta'(r)}(u + \varphi(x)) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega; \end{cases}$$

(c_6) *one has*

$$\eta'(r) = \frac{1}{\psi^{-1}(r)}$$

for all $r \in]0, \delta[$.

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