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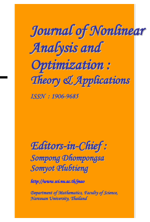

COAPPROXIMATION IN PROBABILISTIC 2-NORMED SPACES

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ABSTRACT. In this article, we studied the best coapproximation in probabilistic 2-normed spaces. We defined the best coapproximation on these spaces and generalized some definitions such as set of best coapproximation, P_b-coproximinal set and P_b-coapproximately compact and orthogonality relative to any set and proved some theorems about them.

KEYWORDS : Probabilistic 2-normed spaces; P_b-best coapproximation; P_b-coproximinal; P_b-coChebyshev.
AMS Subject Classification: 54E70, 46S50

1. INTRODUCTION

In [5], K. Menger introduced the notion of probabilistic metric spaces. The idea of K. Menger was to use distribution function in stead of non negative real numbers as values of the metric. The concept of probabilistic normed spaces (briefly, PN-spaces) was introduced by A. N. Sertnev in 1963, [6].
In [7],[4] the authors have introduced the concept of p-best approximation in probabilistic normed and 2-normed spaces. The main aim of this paper is to investigate another kind of best approximation that called best coapproximation in probabilistic 2-normed spaces. In the sequel after an introduction to probabilistic 2-normed spaces, we define the concept of best coapproximation in probabilistic 2-normed space and generalized some definitions such as set of best coapproximation, coproximinal set and coapproximatively compact set.
Chang et al. [1] defined some notions as follows:

A distance distribution function (briefly, $d.d.f.$), is a function F defined from extended interval $[0, +\infty]$ into the unit interval $I = [0, 1]$, that, is non decreasing and left continuous on $(0, +\infty)$ such that $F(0) = 0$ and $F(+\infty) = 1$. The family of all $d.d.f{,}s$ will be denoted by Δ^+ and we denote

$$D^+ = \{F \in \Delta^+ \mid lim_{t\to\infty} F(t) = 1\}.$$

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By setting $F \leq G$ when ever $F(t) \leq G(t)$, for all $t \in \mathbb{R}^+$, one introduces a natural ordering in D^+. If $a \in \mathbb{R}^+$ then H will be an element of D^+, defined by $H(t) = 0$ if $t \leq 0$ and $H(t) = 1$ if $t > 0$. It is obvious that $H \geq F$ if $t > 0$ for all $F \in D^+$.
A t-norm T is a two place function $T : I \times I \longrightarrow I$ which is associative, commutative, non decreasing in each place and such that $T(a, 1) = a$, for all $a \in [0, 1]$.
Let T be a t-norm and T^* is the function given by

$$T^*(x, y) = 1 - T(1 - x, 1 - y)$$

for all $x, y \in I$. Then T^* is the t-conorm of T.
A triangle function is a mapping $\tau : \Delta^+ \times \Delta^+ \longrightarrow \Delta^+$ which is associative, commutative, non decreasing and for which H is the identity, that is, $\tau(H, F) = F$, for every $F \in D^+$.

Definition 1.1. Let V be a linear space of dimension greater than 1 over filed $\mathbb{R}$ of real numbers, τ a triangle function, and let $\mathcal{F}$ be a mapping from $V \times V$ into D^+ satisfying the following conditions:
a) $F_{x,y} = H$ if and only if x and y are linearly dependent vectors.
b) $F_{x,y} \neq H$ if and only if x and y are linearly independent vectors.
c) $F_{x,y} = F_{y,x}$, for all $x, y \in V$.
d) $F_{\alpha x,y} = F_{x,y}(\frac{t}{|\alpha|})$, for every $t > 0$, $\alpha \neq 0$, $\alpha \in \mathbb{R}$ and $x, y \in V$.
e) $F_{x+y,z} \geq \tau(F_{x,z}, F_{y,z})$ for all $x, y, z \in V$.
Then $\mathcal{F}$ is called a probabilistic 2-norm on V and $(V, \mathcal{F}, \tau)$is called a probabilistic 2-normed space(briefly P2N- Space), and $\mathcal{F}$ is a strong probabilistic 2-norm if $b \in V$ and $t > 0$, $x \longrightarrow F_{x,b}(t)$ is a continuous map on V.
If the triangle inequality (e)is formulated under a t-norm T:
(f) $F_{x+y,z}(t_1 + t_2) \geq T(F_{x,z}(t_1), F_{y,z}(t_2)), for\ all\ x, y, z \in V,\ t_1, t_2 \in \mathbb{R}^+,$
then the triple $(V, \mathcal{F}, T)$ is called a Menger probabilistic 2-normed space.
If T is a left continuous t-norm and τ_T is the associated triangle function, then the inequalities (e)and(f)are equivalent.

Remark 1.2. It is easy to check that every 2-normed space $(V, \|.,.\|)$ can be made a probabilistic 2-normed space, in a natural way, by setting $F_{x,y} = H(t - \|x, y\|)$, for every $x, y \in V$, $t \in \mathbb{R}^+$ and $T = Min$.

Definition 1.3. Let $G \in \Delta^+$ be different from H, let $(V, \|.,.\|)$ be a 2-normed space. Define be a mapping from $\mathcal{F} : V \times V \to \Delta^+$, by $F_{x,y} = H$, if x and y are linearly dependent and

$$F_{x,y}(t) := G(\frac{t}{\|x, y\|})\ (t > 0)$$

when x and y are linearly independent. The pair $(V, \mathcal{F})$ is called the simple space generated by $(V, \|.,.\|)$ and G.

Let $(V, \|.,.\|)$ be a 2-normed space. Define for each $b \in V$, $\tau(F, G)(x) = F(x).G(x)$ for every $F, G \in \Delta^+$ and $F^{\|.,.\|}_{x,b}(t) = \frac{t}{(t+\|x,b\|)}$ for every $x \in V$, then $F^{\|.,.\|}$ is a $P - 2\ norm$ which is called the standard $P - 2\ norm$ induced by $\|.,.\|$.
I. Golet in [3] proved that if $(V, \mathcal{F}, \tau)$ is a probabilistic 2-normed space and $\mathcal{A}$ is the family of all finite and non-empty subsets of the linear space V. For every $A \in \mathcal{A}$, $\varepsilon > 0$ and $\lambda \in (0, 1), (V, \mathcal{F}, \tau)$ is a Hausdorff topological space in the topology τ induced by the family of (ε, λ)-neighborhoods of x_0 vector:

$$\nu_{x_0} = \{N_{x_0}(\varepsilon, \lambda, A) : \varepsilon > 0\ ,\ \lambda \in (0, 1)\ , A \in \mathcal{A}\}$$

Where

$$N_{x_0}(\varepsilon, \lambda, A) = \{x \in V : F_{x_0 - x,a}(\varepsilon) > 1 - \lambda\ , a \in A\}$$

Under a continuous triangle function τ such that $\tau \geq \tau_{T_m}$, where $T_m(a,b) = \max\{a+b-1,0\}$.

2. P_b-BEST COAPPROXIMATION IN PROBABILISTIC 2-NORMED SPACE

Definition 2.1. Let A be a nonempty subset of a P2N-space $(V,\mathcal{F})$. For $t>0$ and $b \in V$, an element $a_0 \in A$ is called a P_b-best coapproximation to $x \in V$ from A if for every $a \in A$,

$$F_{a_0-a,b}(t) \geq F_{x-a,b}(t).$$

The set of all such elements a_0 that called a P_b-best coapproximation to $x \in V$, is denoted by $R^t_{A,b}(x)$, i.e.,

$$R^t_{A,b}(x) = \{a_0 \in A : F_{a_0-a,b}(t) \geq F_{x-a,b}(t) \text{ for all } a \in A,\ t>0\}.$$

Putting

$$\breve{A}_b = \{x \in V : F_{a,b}(t) \geq F_{a-x,b}(t) \text{ for all } a \in A,\ t>0\} = (R^t_{A,b})^{-1}(\{0\}),$$

it is clear $a_0 \in R^t_{A,b}(x)$ if and only if $x-a_0 \in \breve{A}_b$.

Definition 2.2. Let $(V,\mathcal{F})$ be a P2N-space. For $t>0$ and $b \in V$, the nonempty subset $A \subset V$ is called P_b-coproximinal set if $R^t_{A,b}(x)$ is non-void for every $x \in V$ and A is called P_b-coChebyshev set if for every $x \in V$ the set $R^t_{A,b}(x)$ contains exactly one element.

Remark 2.3. Let A be a nonempty subset of a P2N-space $(V,\mathcal{F})$, and $\{x_n\}$ be a sequence of V.

(i)Then the sequence $\{x_n\}$ is said to be P_b-convergent to $x \in V$ and denoted by $x_n \xrightarrow{P_b} x$, if $\lim_{n\to\infty} F_{x_n-x,b}(t)=1$, for all $x \in V$ and $t>0$.

(ii)The set A is closed if and only if, whenever $\{a_n\}$ is a sequence of points in A converging to $x \in V$, then x is also in A.

Theorem 2.4. *Let A be a nonempty subset of a P2N-space $(V,\mathcal{F})$. Then for $t>0$:*

(i) $R^t_{A+y,b}(x+y) = R^t_{A,b}(x)+y$, for every $x,y \in V$.

(ii) $R^{|\alpha|t}_{\alpha A,b}(\alpha x) = \alpha R^t_{A,b}(x)$, for every $x \in V$ and any scaler $\alpha \in \mathbb{R}\backslash\{0\}$.

(iii) A is P_b-coproxminal (respectively P_b-coChebyshev) if and only if $A+y$ is P_b-coproxminal (respectively P_b-coChebyshev) for every $y \in V$.

Proof. (i) For any $x,y \in V$, $t>0$ and $b \in V$, let $a_0 \in R^t_{A+y,b}(x+y)$ if and only if, $F_{a_0-(a+y),b}(t) \geq F_{x+y-(a+y),b}(t)$ for all $(a+y) \in A+y$ if and only if, $F_{(a_0-y)-a,b}(t) \geq F_{x-a,b}(t)$ for all $a \in A$ if and only if, $(a_0-y) \in R^t_{A,b}(x)$ i.e., $a_0 \in R^t_{A,b}(x)+y$.

(ii) Let $a_0 \in R^{|\alpha|t}_{\alpha A,b}(\alpha x)$, for any $x \in V$, $t>0$ and $\alpha \in \mathbb{R}\backslash\{0\}$ if and only, $F_{a_0-\alpha a,b}(\mid \alpha \mid t) \geq F_{\alpha x-\alpha a,b}(\mid \alpha \mid t)$ for all $a \in A$ if and only, $F_{\frac{1}{\alpha}a_0-a,b}(t) \geq F_{x-a,b}(t)$ if and only, $\frac{1}{\alpha}a_0 \in R^t_{A,b}(x)$ if and only, $a_0 \in \alpha R^t_{A,b}(x)$. Therefore, $R^{|\alpha|t}_{\alpha A,b}(\alpha x) = \alpha R^t_{A,b}(x)$.

(iii) Is an immediate consequence of (i). □

Theorem 2.5. *Let $(V,\mathcal{F},\tau)$ be a Menger P2N-space and A be a convex subset of V. Then for $t>0$, $b \in V$ and $x \in V$, $R^t_{A,b}(x)$ is a convex subset of A (for $R^t_{A,b}(x) \neq \emptyset$).*

Proof. Let $a_1,a_2 \in R^t_{A,b}(x)$, $t>0$, $b \in V$ and $x \in V$, then

$$F_{a-a_1,b}(t) \geq F_{x-a,b}(t) \text{ and } F_{a-a_2,b}(t) \geq F_{x-a,b}(t) \text{ for all } a \in A.$$

For $\lambda \in (0,1)$:

$$F_{a-(\lambda a_1+(1-\lambda)a_2),b}(t) = F_{\lambda a-\lambda a_1+a-\lambda a-a_2+\lambda a_2,b}(t)$$

$$= F_{\lambda(a-a_1)+(1-\lambda)(a-a_2),b}(t)$$
$$\geq \tau\big(F_{a-a_1,b}(\frac{\lambda t}{\lambda}), F_{a-a_2,b}(\frac{(1-\lambda)t}{(1-\lambda)})\big)$$
$$\geq \tau\big(F_{x-a,b}(t), F_{x-a}(t)\big) = F_{x-a,b}(t),$$

so for each $\lambda \in (0,1)$, we have

$$F_{a-(\lambda a_1+(1-\lambda)a_2),b}(t) \geq F_{x-a,b}(t),$$

then $\lambda a_1 + (1-\lambda)a_2 \in R^t_{A,b}(x)$.
Hence $R^t_{A,b}(x)$ is a convex. □

Theorem 2.6. *Let $(V, \mathcal{F}, \tau)$ be a Menger P2N-space and A be a subset of V and $b \in V$. If $a_0 \in R^t_{A,b}(x)$ and $(1-\lambda)x + \lambda a_0 \in A$, for $x \in V$ and every scaler $\lambda \neq 0$, then $(1-\lambda)x + \lambda a_0 \in R^t_{A,b}(x)$.*

Proof. Let $a_0 \in R^t_{A,b}(x)$, $t > 0$, $b \in V$ and $x \in V$, then $F_{a-a_0,b}(t) \geq F_{x-a,b}(t)$ for all $a \in A$.
Then for, for $\lambda \neq 0$:

$$\begin{aligned} F_{a-[(1-\lambda)x+\lambda a_0],b}(t) &= F_{a-(1-\lambda)x-\lambda a+\lambda a-\lambda a_0,b}(t) \\ &= F_{(1-\lambda)a-(1-\lambda)x+\lambda(a-a_0),b}(t) \\ &= F_{(1-\lambda)(a-x)+\lambda(a-a_0),b}(t) \\ &\geq \tau\big(F_{a-x,b}(\frac{(1-\lambda)t}{(1-\lambda)}), F_{a-a_0,b}(\frac{\lambda t}{\lambda})\big) \\ &\geq \tau\big(F_{a-x,b}(t), F_{x-a,b}(t)\big) = F_{x-a,b}(t), \end{aligned}$$

for all $a \in A$, thus $(1-\lambda)x + \lambda a_0 \in R^t_{A,b}(x)$. □

Example 2.7. Let $V = \mathbb{R}^2$. Define $\mathcal{F} : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathcal{D}^+$ as

$$F_{(x_1,x_2),(y_1,y_2)}(t) = \big(exp(|x_1y_2 - x_2y_1|/t)\big)^{-1}.$$

Then $(V, \mathcal{F}, \tau)$ is a Menger P2N-space where $\tau(F(t), G(t)) = F(t).G(t)$ for every F and G in $\mathcal{D}^+$. Let $A = \{(x_1, x_2) \in \mathbb{R}^2 \mid -1 \leq x_1 \leq 1,\ 0 \leq x_2 \leq |x_1|\}$ and $x = (0,3), b = (0,2)$. Then for every $t > 0$, $(1,1), (-1,1) \in R^t_{A,b}(0,3)$.

Theorem 2.8. *For $t > 0$ and $b \in V$, let A be a P_b-coproximinal subspace of a P2N-space $(V, \mathcal{F})$. Then*

(1) if $\breve{A}_b$ is a compact set then $R^t_{A,b}(x)$ is compact, for every $x \in V$.

(2) if $\breve{A}_b$ is a close set then $R^t_{A,b}(x)$ is close, for every $x \in V$.

Proof. (1) Suppose $x \in V$ and $\{a_n\}$ is a sequence in $R^t_{A,b}(x)$. Since $x - a_n \in \breve{A}_b$ and $\breve{A}_b$ is a compact set, there is a subsequence $\{x - a_{n_k}\}$ that convergence to $u_0 \in \breve{A}_b$. Since $x - u_0 = a_0$, therefore $a_0 \in R^t_{A,b}(x)$.
(2) It is clear. □

The following lemma shows that the P_b-best coapproximation in probabilistic 2-normed spaces is a generalization of best coapproximation in 2-normed spaces.

Lemma 2.9. *Let $(V, \|.,.\|)$ be a 2-normed space and $F^{\|.,.\|}$ be the induced probabilistic 2-norm. Then for $b \in V$, $y_0 \in A$ is a best coapproximation to $x \in V$ in the 2-normed linear space if and only if y_0 is a P_b-best coapproximation to x in the induced probabilistic 2-normed linear space $(V, \mathcal{F}^{\|.,.\|}, \tau)$,* *Proof.* For $b \in V$, since y_0 is a best coapproximation to $x \in V$, we have $\{\|y-y_0, b\| \leq \|x-y, b\| \ ; \ \forall y \in A\}$ if and only if $\{\frac{t}{t+\|y-y_0,b\|} \geq \frac{t}{t+\|x-y,b\|} \ ; \ \forall y \in A\}$ if and only if $\{F^{\|.,.\|}_{y-y_0,b}(t) \geq F^{\|.,.\|}_{x-y,b}(t) \ ; \ \forall y \in A\}$ if and only if $y_0 \in R^t_{A,b}(x)$. □

Definition 2.10. For $t > 0$ and $b \in V$, let $(V, \mathcal{F}, \tau)$ be a Menger P2N-space and A be a subset of V. An element $x \in V$ is said to be b-orthogonal to an element $y \in V$, and we denote $x \perp^b y$, if $F_{x+\lambda y,b}(t) \leq F_{x,b}(t)$ for all scalar $\lambda \in \mathbb{R}$, $\lambda \neq 0$ and $t > 0$.

Also, An element $x \in V$ is said to be $b-$orthogonal to A, and we denote $x \perp^b A$, if $x \perp^b y$, for all $y \in A$.

Theorem 2.11. *For $t > 0$ and $b \in V$, let $(V, \mathcal{F}, \tau)$ be a Menger P2N-space and A be a subset of V. Then for $x \in V$, $y_0 \in R^t_{A,b}(x)$ if and only if $A \perp^b x - y_0$.*

Proof. Suppose $x \in V$ and $A \perp^b x - y_0$. Then $F_{a+\lambda(x-y_0),b}(t) \leq F_{a,b}(t)$ for all $a \in A$ and all scalar $\lambda \in \mathbb{R}$, $\lambda \neq 0$ and $t > 0$, if and only if $F_{x-y_0+\lambda^{-1}a,b}(\frac{t}{|\lambda|}) \leq F_{\lambda^{-1}a,b}(\frac{t}{|\lambda|})$, if and only if $F_{x-y_0+a',b}(\frac{t}{|\lambda|}) \leq F_{a',b}(\frac{t}{|\lambda|})$ and for $a^{'} = \lambda^{-1}a$ and $y_0 - a^{'} = a^{''}$, if and only if $F_{x-a'',b}(\frac{t}{|\lambda|}) \leq F_{y_0-a'',b}(\frac{t}{|\lambda|})$ for every $\lambda \neq 0$, if and only if $F_{a-y_0,b}(t) \geq F_{x-a,b}(t)$ for all $a \in A$ and for each $t > 0$, if and only if $y_0 \in R^t_{A,b}(x)$. □

Remark 2.12. For $t > 0$ and $b \in V$, let $(V, \mathcal{F}, \tau)$ be a Menger P2N-space and A be a subset of V.

$$(R^t_{A,b})^{-1}(\{0\}) = \{x \in V : F_{a,b}(t) \geq F_{a-x,b}(t) \text{ for all } a \in A,\ t > 0\} = \{x \in V : A \perp^b x\},$$

$$\breve{A}_b = \{x \in V : A \perp^b x\}.$$

Theorem 2.13. *Let A be subspace of a Menger P2N-space $(V, \mathcal{F}, \tau)$, then $\breve{A}_b \bigcap A = \{0\}$.*

Proof. Let $a \in \breve{A}_b \bigcap A$, we show that $a = 0$. To see this, we have $a \in \breve{A}_b$, then $A \perp^b a$ and $a \in A$, this implies that $h \perp^b a$ for all $h \in A$.
Therefore, $F_{h+\lambda a,b}(t) \leq F_{h,b}(t)$ for all $h \in A,\ t > 0$, and all scaler λ.
Now, if we choose $\lambda = -\frac{1}{3}$ and $h = a$, then $F_{a-\frac{1}{3}a,b}(t) \leq F_{a,b}(t)$, and so, $F_{\frac{2}{3}a,b}(t) = F_{a,b}(\frac{3}{2}t) \leq F_{a,b}(t)$, and hence, $a = 0$, i.e., $\breve{A}_b \bigcap A \subseteq \{0\}$. But $\{0\} \subseteq \breve{A}_b \bigcap A$, together, we get $\breve{A}_b \bigcap A = \{0\}$. □

Theorem 2.14. *For $t > 0$ and $b \in V$, let A be a P_b-coproximinal subspace of a Menger P2N space $(V, \mathcal{F}, \tau)$. If $\breve{A}_b$ is a convex set, then A is P_b- coChebyshev, for every $x \in V$.*

Proof. Suppose $t > 0$, $x \in V$ and $a_1, a_2 \in R^t_{A,b}(x)$; then $x - a_1, x - a_2 \in \breve{A}_b$. Put $\breve{a}_1 = x - a_1$ and $\breve{a}_2 = x - a_2$ and let us have $x = a_1 + \breve{a}_1 = a_2 + \breve{a}_2$. Since $\frac{1}{2}(\breve{a}_1 - \breve{a}_2) \in \breve{A}_b$, it follows that $a_1 - a_2 \in \breve{A}_b \cap A = \{0\}$; then $a_1 = a_2$. □

Definition 2.15. For $t > 0$ and $b \in V$, let $(V, \mathcal{F}, \tau)$ be a Menger P2N-space, A and H be subsets of V. Define: $R^t_{A,b}(H) = \bigcup_{h \in H} R^t_{A,b}(h)$.

Theorem 2.16. *For $t > 0$ and $b \in V$, let $(V, \mathcal{F}, \tau)$ be a Menger P2N-space, A and $A^{'}$ be subspaces of V, such that $A \subseteq A^{'}$, and let $x \in V$. Then:*

$$R^t_{A,b}(R^t_{A',b}(x)) \subseteq R^t_{A,b}(x).$$

Proof. Suppose $a_0 \in R^t_{A,b}(R^t_{A',b}(x))$, then $a_0 \in R^t_{A,b}(a_0')$ for $a_0' \in R^t_{A',b}(x)$, so $A' \perp^b (x - a_0')$, and $A \perp^b (a_0' - a_0)$. Thus, $F_{a'+\lambda(x-a_0'),b}(t) \leq F_{a',b}(t)$ for all $\lambda \in \mathbb{R}$ and $a' \in A'$, and $F_{a+\lambda(a_0'-a_0),b}(t) \leq F_{a,b}(t)$ for all $\lambda \in \mathbb{R}$ and $a \in A$. Now since, $a + \lambda(a_0' - a_0) \in A'$ for $\lambda \in \mathbb{R}$ and $a \in A \subset A'$, therefore,

$$F_{a+\lambda(x-a_0),b}(t) = F_{a+\lambda(a_0'-a_0)+\lambda(x-a_0'),b}(t) \leq F_{a+\lambda(a_0'-a_0),b}(t) \leq F_{a,b}(t),$$

since $F_{a+\lambda(x-a_0),b}(t) \leq F_{a,b}(t)$, so, $a \perp^b (x - a_0)$ for all $a \in A$, then $A \perp^b (x - a_0)$, i.e., $a_0 \in R^t_{A,b}(x)$. Hence $R^t_{A,b}(R^t_{A',b}(x)) \subseteq R^t_{A,b}(x)$. □

Corollary 2.17. *For $t > 0$ and $b \in V$, let $(V, \mathcal{F}, \tau)$ be a Menger P2N-space, and A be subspace of V. Then $R^t_{A,b}(x) = A \bigcap (x - \breve{A}_b)$.*

Proof. Let $a_0 \in A \bigcap (x - \breve{A}_b)$, if and only if $a_0 \in A$, and $a_0 \in (x - \breve{A}_b)$, if and only if $a_0 \in A$, and $a_0 = x - \breve{a}$, where $\breve{a} \in \breve{A}_b$, if and only if $a_0 \in A$, and $\breve{a} = x - a_0 \in \breve{A}$, if and only if $a_0 \in R^t_{A,b}(x)$. Therefore, $R^t_{A,b}(x) = A \bigcap (x - \breve{A}_b)$. □

Theorem 2.18. *Let A be subspace of a Menger P2N-space $(V, \mathcal{F}, \tau)$, then*
(1) A is a P_b-coproximinal subspace if and only if $V = A + \breve{A}_b$.
(2) A is a P_b-coChebyshev subspace if and only if $V = A \oplus \breve{A}_b$.

Proof. (1)(⇒)Let $t > 0$, assume that A is P_b-coproximinal, and let $x \in V$ and $a_0 \in R^t_{A,b}(x)$. Then, $x - a_0 \in \breve{A}_b$. Now, $x = a_0 + (x - a_0) \in A + \breve{A}_b$. Hence, $V = A + \breve{A}_b$.

(⇐) Let $t > 0$, $V = A + \breve{A}_b = \{a + y : a \in A, y \in \breve{A}_b\}$, and $x \in V$. Then $x = a_0 + y$, where $a_0 \in A$, $y \in \breve{A}_b$. Since $y \in \breve{A}_b = R^{-t}_{A,b}(0)$, then $0 \in R^t_{A,b}(y)$. Since $x = a_0 + y$, then $y = x - a_0$, so $R^t_{A,b}(y) = R^t_{A,b}(x - a_0)$, this implies that $0 \in R^t_{A,b}(y) = R^t_{A,b}(x - a_0)$.
Then $F_{0-(x-a_0),b}(t) \geq F_{a-(x-a_0),b}(t)$, so $F_{a_0-x,b}(t) \geq F_{(a+a_0)-x,b}(t)$ where $(a+a_0) \in A$; hence $a_0 \in R^t_{A,b}(x)$. Therefore A is P_b-coproximinal.

(2)(⇒) Suppose that A is P_b-coChebyshev subspace and $x \in V$, $x = a_1 + \breve{a}_1 = a_2 + \breve{a}_2$, where $a_1, a_2 \in A$ and $\breve{a}_1, \breve{a}_2 \in \breve{A}_b$.
We show that $a_1 = a_2$, and $\breve{a}_1 = \breve{a}_2$, since $x = a_1 + \breve{a}_1 = a_2 + \breve{a}_2$, then $x - a_1 = \breve{a}_1, x - a_2 = \breve{a}_2 \in \breve{A}$, this implies that $a_1, a_2 \in R^t_{A,b}(x)$.
Therefore, $a_1 = a_2$ because A is P_b-coChebyshev, it follows that $\breve{a}_1 = \breve{a}_2$. Thus $V = A \oplus \breve{A}_b$.

(⇐) Let $V = A \oplus \breve{A}_b$ and suppose for $x \in V$, there exist $a_1, a_2 \in R^t_{A,b}(x)$. We show $a_1 = a_2$.
Since $a_1, a_2 \in R^t_{A,b}(x)$, then $x - a_1, x - a_2 \in \breve{A}_b$ and therefore, $x = a_1 + \breve{a}_1 = a_2 + \breve{a}_2$, where $\breve{a}_1 = x - a_1$ *and* $\breve{a}_2 = x - a_2$. Since $V = A \oplus \breve{A}$, then $a_1 = a_2$ and $\breve{a}_1 = \breve{a}_2$. Hence A is P_b-coChebyshev. □

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