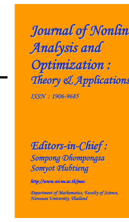
**J**ournal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 3, No. 1, (2012), 79-84
**ISSN : 1906-9605**
**http://www.math.sci.nu.ac.th**




# ASYMPTOTIC BEHAVIOR OF INFINITE PRODUCTS OF PROJECTION AND NONEXPANSIVE OPERATORS WITH COMPUTATIONAL ERRORS

EVGENIY PUSTYLNIK*, SIMEON REICH AND ALEXANDER J. ZASLAVSKI

Department of Mathematics, The Technion–Israel Institute of Technology, 32000 Haifa, Israel

**ABSTRACT.** We study the asymptotic behavior of infinite products of orthogonal projections and other (possibly nonlinear) nonexpansive operators in Hilbert space in the presence of computational errors.

**KEYWORDS** : Hilbert space; Infinite product; Nonexpansive operator; Orthogonal projection.
**AMS Subject Classification**: 46C05, 47H09, 47H10.

## 1. INTRODUCTION

Consider $m$ closed linear subspaces $S_1, S_2, \dots, S_m$ of a given Hilbert space $H$ and let $S$ denote their intersection. Let the infinite product $\prod_{j=1}^{\infty} P_j := \cdots P_3 P_2 P_1$ only consist of orthogonal projections $P_{S_k},\ 1 \le k \le m$, onto these subspaces. We are concerned with the asymptotic behavior of the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_n = \prod_{j=1}^{n} P_j x_0 = P_n P_{n-1} ... P_1 x_0,\ n = 1, 2, \dots .$$

Denoting the norm of $H$ by $||\cdot||$, we recall that the classical theorems of J. von Neumann [10] and I. Halperin [9] declare that, for any $x_0 \in H$,

$$\lim_{n \longrightarrow \infty} ||(P_{S_m} P_{S_{m-1}} \cdots P_{S_1})^n x_0 - P_S x_0|| = 0. \tag{1.1}$$

We observe that the iterative process here is strongly cyclical and this condition is, in fact, essential for the proof of (1.1). The convergence in (1.1) may not be uniform (on bounded subsets of initial points) and these theorems do not provide any rate of convergence.

Except for some earlier partial results, general necessary and sufficient conditions for uniform convergence in (1.1) were found much later (see [6] for $m = 2$ and [2] for the general case). In addition, some estimates of the rate of this convergence were also obtained, mainly by using the notion of angles between subspaces [8].

* *Corresponding author.*
*Email address* : evg@tx.technion.ac.il (Evgeniy Pustylnik), sreich@tx.technion.ac.il (Simeon Reich) and ajzasl@tx.technion.ac.il (Alexander J. Zaslavski).
*Article history* : Received 5 January 2012. Accepted 18 March 2012.

Traditionally, these estimates were asserted for cyclical products, although this restriction was not as essential as it was for proving (1.1) in [9]. As a matter of fact, these estimates were stated for one cycle, but then immediately generalized to the power $n$. Consequently, the possibility of concatenating different fragmentary products of projections into an infinite one was not considered.

We have recently [15–17] used a geometric approach (in particular, angles between subspaces) to establish useful estimates for proving convergence of infinite products involving not only orthogonal projections, but also other (possibly nonlinear) nonexpansive operators in Hilbert space. In [18] we provide sufficient conditions for the strong and uniform (on bounded subsets of initial points) convergence of such infinite products by applying new estimates of the inclination [1] of a finite tuple of closed linear subspaces.

In view of the diverse applications of infinite products (see, for instance, [7] and the references therein), it is natural to ask if these results continue to hold in the presence of computational errors. In the present paper we give affirmative answers to this question. Our main results, Theorems 2.1 and 2.2, are formulated in Section 2. Their proofs are given in Section 3.

Previous studies concerning inexact powers and infinite products of operators can be found, for example, in [3–5, 11–14, 19].

## 2. MAIN RESULTS

For each $x \in H$ and each $B \subset H$, set

$$\rho(x, B) = \inf\{||x - y|| : \ y \in B\}.$$

We consider an iterative process, presented as an infinite product $\prod_{i=1}^{\infty} A_i \mathcal{P}_i$, where all $A_i$ are quasi-nonexpansive, possibly nonlinear, operators of arbitrary nature and each $\mathcal{P}_i$ is a finite product of all the projections $P_{S_1}, P_{S_2}, \dots, P_{S_m}$ in any order and amount (that is, with possible repetitions). Here $S_1, S_2, \dots, S_m$ are assumed to be closed linear subspaces of a given Hilbert space $H$. By $S$ we denote the intersection of $S_1, S_2, \dots, S_m$; the case $S = \{0\}$ is permitted as well. We assume that all the subspaces $S_j$ are different. Recall that the principal question studied in this paper concerns the asymptotic behavior of the sequence $\{x_n\}_{n=0}^{\infty}$ defined by $x_n = \prod_{i=1}^{n} A_i \mathcal{P}_i x_0, \ n = 1, 2, \dots$, in the presence of computational errors.

We are now ready to state our two main results.

**Theorem 2.1.** *Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of self-mappings of a Hilbert space $H$. Assume that for all integers $n \geq 1$,*

$$A_n(S) \subset S \tag{2.1}$$

*and*

$$||A_n y - A_n x|| \leq ||y - x|| \text{ for all } y \in H \text{ and all } x \in S. \tag{2.2}$$

*Assume further that for all integers $n \geq 1$ and all $x \in H$,*

$$||\mathcal{P}_n x - P_S x|| \leq q_n ||x - P_S x|| \tag{2.3}$$

*with the factors $q_n \in (0, 1]$ satisfying*

$$\lim_{n \longrightarrow \infty} \prod_{i=1}^{n} q_i = 0 \tag{2.4}$$

*and*

$$M_0 := \sup\{1 + \sum_{p=2}^{k} \prod_{i=p}^{k} q_i : \ k = 2, 3, \dots\} < \infty. \tag{2.5}$$

*Let the positive numbers $M, \epsilon, \delta$ and a natural number $n_0$ satisfy*

$$\prod_{i=1}^{n_0} q_i M \le \epsilon/2 \tag{2.6}$$

*and*

$$\delta M_0 \le \epsilon/2. \tag{2.7}$$

*Finally, assume that $\{x_i\}_{i=0}^{\infty} \subset H$,*

$$||x_0|| \le M, \tag{2.8}$$

*and that for any integer $n \ge 1$,*

$$||x_n - A_n \mathcal{P}_n x_{n-1}|| \le \delta. \tag{2.9}$$

*Then*

$$\rho(x_i, S) \le \epsilon \text{ for all integers } i \ge n_0.$$

**Theorem 2.2.** *Assume that (2.1) and (2.2) hold for all integers $n \ge 1$, and that (2.3) holds for all integers $n \ge 1$ and all points $x \in H$ with the factors $q_n \in (0, 1]$ satisfying both (2.4) and (2.5). Assume further that $\{\delta_i\}_{i=1}^{\infty} \subset (0, \infty)$ and*

$$\lim_{i \longrightarrow \infty} \delta_i = 0. \tag{2.10}$$

*Let $\epsilon, M > 0$ be given. Then there exists an integer $n_1 \ge 1$ such that for each sequence $\{x_n\}_{n=0}^{\infty} \subset H$ satisfying*

$$||x_0|| \le M \tag{2.11}$$

*and*

$$||x_n - A_n \mathcal{P}_n x_{n-1}|| \le \delta_n \text{ for all integers } n \ge 1, \tag{2.12}$$

*the following inequality holds:*

$$\rho(x_n, S) \le \epsilon \text{ for all integers } n \ge n_1. \tag{2.13}$$

Before proceeding to the proofs of these theorems in Section 3, we observe that both (2.4) and (2.5) clearly hold if for all $i = 1, 2, \dots,$

$$q_i \le q < 1$$

for some constant $q$.

More generally, both (2.4) and (2.5) hold if there are a real number $q \in (0, 1)$ and a natural number $p$ such that

$$q_{ip} \le q \text{ for all natural numbers } i.$$

In this connection, recall that the number

$$l(S_1, S_2, \dots, S_m) = \inf_{x \notin S} \max_{1 \le j \le m} \rho(x, S_j) \rho(x, S)^{-1}$$

is called the inclination of the $m$-tuple $(S_1, S_2, \dots, S_m)$. Clearly, $0 \le l \le 1$.

By [1], for any set of integers $\{i_1, \dots, i_N\} = \{1, 2, \dots, m\}$ and any $x \in H$,

$$||P_{S_{i_N}} P_{S_{i_{N-1}}} \cdots P_{S_{i_1}} x - P_S x|| \le (1 - l^2 N^{-2})^{1/2} ||x - P_S x||.$$

Thus (2.4) and (2.5) hold if there is a natural number $p$ such that

$$\sup\{N_{ip} : \ i = 1, 2, \dots\} < \infty,$$

where $N_k$ is the number of operators in the product $\mathcal{P}_k$.

## 3. PROOFS OF THEOREMS 2.1 AND 2.2

We begin with the following lemma.

**Lemma 3.1.** *Assume that both (2.1) and (2.2) hold for each integer $n \geq 1$, (2.3) holds for all integers $n \geq 1$ and all points $x \in H$, $\delta$ is a positive number, and the sequence $\{x_n\}_{n=0}^{\infty} \subset X$ satisfies*

$$||x_n - A_n \mathcal{P}_n x_{n-1}|| \leq \delta \tag{3.1}$$

*for all integers $n \geq 1$. Then for any integer $k \geq 2$,*

$$\rho(x_k, S) \leq (\prod_{i=1}^{k} q_i)\rho(x_0, S) + \delta(1 + \sum_{p=2}^{k}(\prod_{i=p}^{k} q_i)). \tag{3.2}$$

*Proof.* In view of (3.1), for any integer $n \geq 1$,

$$\begin{aligned}\rho(x_n, S) &\leq ||x_n - A_n \mathcal{P}_n x_{n-1}|| + \rho(A_n \mathcal{P}_n x_{n-1}, S)\\ &\leq \delta + ||A_n \mathcal{P}_n x_{n-1} - A_n P_S x_{n-1}||,\end{aligned}$$

and in view of (2.1) and (2.2),

$$\begin{aligned}\rho(x_n, S) \leq \delta + ||\mathcal{P}_n x_{n-1} - P_S x_{n-1}|| &\leq \delta + q_n ||x_{n-1} - P_S x_{n-1}||\\ &\leq \delta + q_n \rho(x_{n-1}, S).\end{aligned} \tag{3.3}$$

By (3.3),

$$\rho(x_1, S) \leq \delta + q_1 \rho(x_0, S),\ \ \rho(x_2, S) \leq q_1 q_2 \rho(x_0, S) + q_2 \delta + \delta. \tag{3.4}$$

We now show by induction that for any integer $k \geq 2$, inequality (3.2) holds. To this end, we first note that in view of (3.4), inequality (3.2) certainly holds for $k = 2$.

Assume that $n \geq 2$ is an integer and that (3.2) holds for $k = n$. Thus

$$\rho(x_n, S) \leq (\prod_{i=1}^{n} q_i)\rho(x_0, S) + \delta(1 + \sum_{p=2}^{n}(\prod_{i=p}^{n} q_i)). \tag{3.5}$$

By (3.3) and (3.5),

$$\begin{aligned}\rho(x_{n+1}, S) &\leq \delta + q_{n+1}\rho(x_n, S)\\ &\leq \delta + (\prod_{i=1}^{n+1} q_i)\rho(x_0, S) + \delta q_{n+1}(1 + \sum_{p=2}^{n}(\prod_{i=p}^{n} q_i))\\ &= (\prod_{i=1}^{n+1} q_i)\rho(x_0, S) + \delta(1 + q_{n+1} + \sum_{p=2}^{n}(\prod_{i=p}^{n+1} q_i))\\ &= (\prod_{i=1}^{n+1} q_i)\rho(x_0, S) + \delta(1 + \sum_{p=2}^{n+1}(\prod_{i=p}^{n+1} q_i)),\end{aligned}$$

so that (3.2) holds for $k = n+1$. Thus we have shown by induction that inequality (3.2) holds for all integers $k \geq 2$, as claimed. Lemma 3.1 is proved. □

**Completion of the proof of Theorem 2.1.** By Lemma 3.1, (2.8), (2.5), (2.6) and (2.7), for any integer $k \geq n_0$, we have

$$\begin{aligned}\rho(x_k, S) &\leq (\prod_{i=1}^{k} q_i)\rho(x_0, S) + \delta(1 + \sum_{p=2}^{k}(\prod_{i=p}^{k} q_i))\\ &\leq (\prod_{i=1}^{n_0} q_i)M + \delta M_0 \leq \epsilon/2 + \epsilon/2 = \epsilon.\end{aligned}$$

Theorem 2.1 is proved.

**Proof of Theorem 2.2.** Let $M_0$ be defined by (2.5). Choose a positive number $\bar{\delta}$ such that

$$\bar{\delta} M_0 < \epsilon/2. \tag{3.6}$$

By (2.10), there is a natural number $n_0$ such that

$$\delta_i < \bar{\delta} \text{ for all integers } i \geq n_0. \tag{3.7}$$

Choose

$$\widehat{\delta} = \sup\{\delta_i : \ i = 1, 2, \dots\} \tag{3.8}$$

and set

$$M_1 = \widehat{\delta} M_0 + M. \tag{3.9}$$

By (2.4), there is a natural number $n_1 > n_0$ such that

$$M_1 \prod_{i=n_0+1}^{n_1} q_i < \epsilon/2. \tag{3.10}$$

Assume that a sequence $\{x_i\}_{i=0}^{\infty} \subset H$ satisfies (2.11) and (2.12). By (3.8), (2.11), (2.12), Lemma 3.1 and (3.9),

$$\rho(x_{n_0}, S) \leq \prod_{i=1}^{n_0} q_i \rho(x_0, S) + \widehat{\delta} M_0 < M_1. \tag{3.11}$$

For each integer $i \geq 1$, define

$$y_{i-1} = x_{i-1+n_0}, \ \tilde{A}_i = A_{i+n_0}, \ \tilde{\mathcal{P}}_i = \mathcal{P}_{i+n_0}. \tag{3.12}$$

By (3.11), (3.12), (2.12), (2.5), (3.6), (3.7), (3.10) and by Lemma 3.1 applied to $\{y_i\}_{i=0}^{\infty}$, $\{\tilde{A}_i\}_{i=1}^{\infty}$ and $\{\tilde{\mathcal{P}}_i\}_{i=1}^{\infty}$, we have for all integers $k \geq n_1$,

$$\begin{aligned} \rho(x_k, S) &= \rho(y_{k-n_0}, S) \\ &\leq (\prod_{p=n_0+1}^{k} q_p)\rho(y_0, S) + \widehat{\delta}(1 + \sum_{p=n_0+1}^{k} (\prod_{i=p}^{k} q_i)) \\ &\leq \prod_{p=n_0+1}^{n_1} q_p M_1 + \bar{\delta} M_0 < \epsilon/2 + \epsilon/2 = \epsilon, \end{aligned}$$

as asserted. Theorem 2.2 is proved.

**Acknowledgments.** This research was supported by the Israel Science Foundation (Grant No. 647/07), the Fund for the Promotion of Research at the Technion and by the Technion VPR Research Fund.

## References

1. C. Badea, S. Grivaux and V. Müller, The rate of convergence in the method of alternating projections, Preprint, Institute of Mathematics, AS CR, Prague, 2010.
2. H. H. Bauschke and J. M. Borwein, On projection algorithms for solving convex feasibility problems, SIAM Rev. 38 (1996), 367–426.
3. D. Butnariu, S. Reich and A. J. Zaslavski, Convergence to fixed points of inexact orbits of Bregman-monotone and of nonexpansive operators in Banach spaces, Fixed Point Theory and its Applications. Yokohama Publishers, Yokohama, 2006, 11–32.

4. D. Butnariu, S. Reich and A. J. Zaslavski, Asymptotic behavior of inexact orbits for a class of operators in complete metric spaces, J. Appl. Anal. 13 (2007), 1–11.
5. D. Butnariu, S. Reich and A. J. Zaslavski, Stable convergence theorems for infinite products and powers of nonexpansive mappings, Numerical Func. Anal. Optim. 29 (2008), 304–323.
6. F. Deutsch, Best approximation in inner product spaces, Springer. New York, 2001.
7. J. Dye and S. Reich, Random products of nonexpansive mappings, Res. Notes Math. Ser. 244 (1992), Longman Sci. Tech. Harlow, 106–118.
8. K. Friedrichs, On certain inequalities and characteristic value problems for analytic functions and for functions of two variables, Trans. Amer. Math. Soc. 41 (1937), 321–364.
9. I. Halperin, The product of projection operators, Acta Sci. Math.(Szeged) 23 (1962), 96-99.
10. J. von Neumann, On rings of operators, Reduction theory, Ann. of Math. 50 (1949), 401–485.
11. A. M. Ostrowski, The round-off stability of iterations, Z. Angew. Math. Mech. 47 (1967), 77-81.
12. E. Pustylnik, S. Reich and A. J. Zaslavski, Inexact orbits of nonexpansive mappings, Taiwanese J. Math. 12 (2008), 1511–1523.
13. E. Pustylnik, S. Reich and A. J. Zaslavski, Convergence to compact sets of inexact orbits of nonexpansive mappings in Banach and metric spaces, Fixed Point Theory Appl. 2008, Article ID 528614, 1–10.
14. E. Pustylnik, S. Reich and A. J. Zaslavski, Inexact infinite products of nonexpansive mappings, Numerical Func. Anal. Optim. 30 (2009), 632–645.
15. E. Pustylnik, S. Reich and A. J. Zaslavski, Convergence of infinite products of nonexpansive operators in Hilbert space, J. Nonlinear Convex Anal. 11 (2011), 461–474.
16. E. Pustylnik, S. Reich and A. J. Zaslavski, New possibilities regarding the alternating projections method, J. Nonlinear Anal. Optim. 2 (2011), 33–37.
17. E. Pustylnik, S. Reich and A. J. Zaslavski, Convergence of non-cyclic iterative projection methods, J. Math. Anal. Appl. 380 (2011), 759-767.
18. E. Pustylnik, S. Reich and A. J. Zaslavski, Convergence of non-periodic infinite products of orthogonal projections and nonexpansive operators in Hilbert space, J. Approximation Theory 164 (2012), 611–624.
19. S. Reich and A. J. Zaslavski, Inexact powers and infinite products of nonlinear operators, Int. J. Math. Stat. 6 (2010), 89-109.